

Balanced groups and graphs of groups with infinite cyclic edge groups

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Abstract

We give a necessary and sufficient condition for the fundamental group of a finite graph of groups with infinite cyclic edge groups to be acylindrically hyperbolic, from which it follows that a finitely generated group splitting over \mathbb{Z} cannot be simple. We also give a necessary and sufficient condition (when the vertex groups are torsion free) for the fundamental group to be balanced, where a group is said to be balanced if x^m conjugate to x^n implies that $|m| = |n|$ for all infinite order elements x .

1 Introduction

It is well known that a word hyperbolic group cannot contain a Baumslag-Solitar subgroup $BS(m, n)$; indeed these have been called “poison subgroups”. But whereas a group containing, say, $BS(2, 3)$ is already seen to be badly behaved because it contains a non residually finite subgroup and so is itself non residually finite, this need not be the case if a group contains $BS(1, 1) = \mathbb{Z}^2$. On further reflection, the problems seem to arise when we have subgroups $BS(m, n)$ for $|m| \neq |n|$ whereupon we have an infinite order element x with x^m conjugate to x^n . This phenomenon is already an obstruction in various settings, both group theoretic and geometric. One way to think of this is that if we have a “translation length” function τ from a group G to the reals which is invariant on conjugacy classes and such that $\tau(g^n) = |n|\tau(g)$ then any element x with the property above must be sent to zero.

In [17] a group was called unbalanced if there exists an infinite order element x with x^m conjugate to x^n but $|m| \neq |n|$. An unbalanced group need not contain a Baumslag - Solitar subgroup in general but there are conditions which ensure that it does. It was pointed out in this paper that an unbalanced group cannot be subgroup separable and here we also show that an unbalanced group cannot be a subgroup of $GL(n, \mathbb{Z})$ for any n . Indeed we begin in Section 2 by giving various examples of classes of groups that must necessarily be balanced and then in Section 3 we look at unbalanced groups. In order to do this we consider individual elements and say here that an infinite order element x with x^m conjugate to x^n but $|m| \neq |n|$ is unbalanced. We then introduce two straightforward but useful tools in order to determine whether a particular infinite order element a in an arbitrary group G is balanced or not. The first is that of the intersector, consisting of all elements of G that conjugate $\langle a \rangle$ to something which intersects $\langle a \rangle$ non trivially. The point is that for infinite cyclic subgroups this intersector is a subgroup, clearly containing $\langle a \rangle$. The next tool is that of the modular map from the intersector of a to the non zero rationals, sending an element g to m/n if $ga^mg^{-1} = a^n$, which is easily verified to be a well defined homomorphism and which will clearly detect unbalanced elements because it is equivalent to the image of this homomorphism not lying in ± 1 .

Our first applications are to finite graphs of groups where all edge groups are infinite cyclic but with little or no restriction on the vertex groups. In Section 4 we look at when the fundamental group of such a graph of groups is acylindrically hyperbolic. A sufficient condition, with no restriction on edge or vertex groups, was given in [14]. When all edge groups are infinite cyclic but the vertex groups are arbitrary, we use this theorem along with consideration of the intersectors in the vertex groups of the inclusions of the edge subgroups to obtain necessary and sufficient conditions for acylindrical hyperbolicity in Theorem 4.2. An immediate application is that a finitely generated group that splits over \mathbb{Z} cannot be simple.

In the rest of the paper we only deal with torsion free groups. In Section 5 we consider when the fundamental group of a finite graph of groups is relatively hyperbolic. As this implies acylindrical hyperbolicity, we in fact give conditions which ensure that such a group is not relatively hyperbolic with respect to any collection of proper subgroups. For this we allow the vertex groups to be arbitrary torsion free groups and the edge groups to be arbitrary non trivial subgroups of the vertex groups. We then introduce the idea of the malnormal closure of a non trivial subgroup of an arbitrary group

and use this to obtain Corollary 5.5 which states that the fundamental group of a graph of groups as above is not relatively hyperbolic if each vertex group has a cyclic subgroup whose malnormal closure is the whole of this group. For instance this is the case if each vertex group has an infinite soluble normal subgroup.

It is not hard to see that for torsion free groups, being balanced is not preserved in general by taking amalgamated free products and HNN extensions, although it is for free products. In the case where the edge groups are infinite cyclic rather than trivial, we show in Theorem 6.1 using intersectors that an amalgamated free product over \mathbb{Z} of balanced groups will still be balanced, and we give in Theorem 6.4 necessary and sufficient conditions for an HNN extension over \mathbb{Z} . These are then used in Theorem 8.3 to answer the same question for the fundamental group of a finite graph of groups with infinite cyclic edge groups and arbitrary torsion free vertex groups. In order to do this we need to know when two elements both lying in vertex groups have powers that are conjugate in the fundamental group and this is dealt with in Section 7 by introducing the idea of a conjugacy path. This can be thought of as an edge path in the graph that records the fact that successive edge subgroups have conjugates in the intermediate vertex group that intersect non trivially, without having to keep track throughout of the exact powers that occur. This means that our necessary and sufficient condition as to when the fundamental group of a finite graph of groups with infinite cyclic edge groups and arbitrary torsion free vertex groups is balanced in Theorem 8.3 is phrased purely in terms of conjugacy paths that return to their starting element and we prove that there are only finitely many of these paths that need to be checked. One application is that for any such graph of groups that fails this condition, the resulting fundamental group cannot be subgroup separable or embeddable in $GL(n, \mathbb{Z})$ for any n , no matter how well behaved the vertex groups are.

One would also like to say that such a graph of groups fails this condition exactly when the fundamental group contains an unbalanced Baumslag-Solitar subgroup, but we mentioned that some (torsion free) unbalanced groups do not contain these as subgroups. To get round this one could impose some reasonably wide ranging condition on the vertex groups, such as being word hyperbolic or having cohomological dimension 2. We define in Section 9 a condition on a group G which is a substantial generalisation of both of these, called the cohomological condition, which states that any 2-generator subgroup $\langle a, b \rangle$ of G where $\langle a \rangle$ meets $\langle b \rangle$ non trivially has coho-

mological dimension at most 2. We then show in Corollary 9.6 that if all vertex groups are torsion free, balanced and satisfy this condition then the fundamental group is unbalanced if and only if it contains an unbalanced Baumslag - Solitar subgroup.

Our final application in Section 10 is to the characterisation of word hyperbolicity for a graph of groups with word hyperbolic vertex groups and infinite cyclic edge groups. Now the famous Bestvina - Feighn combination theorem in [3] and [4] gives a sufficient condition for a finite graph of groups with word hyperbolic vertex groups to have a word hyperbolic fundamental group and this need not require restriction to infinite cyclic edge groups. In the case where they are infinite cyclic, these papers give necessary and sufficient conditions for an amalgamated free product and HNN extension to be word hyperbolic which can then be applied to determine whether or not the fundamental group of a graph of groups with word hyperbolic vertex groups and infinite cyclic edge groups has this property. However we have not seen in the literature a characterisation of this which is given purely in terms of information that can be directly read off from the graph of groups without consideration of the action of the Bass - Serre tree. (The closest equivalent to this we have come across is Appendix A in [6] which allows one to establish whether the fundamental group of a graph of groups with abelian edge groups has the CSA property.) This can be done by using conjugacy paths as described above, as well as what we call non maximal paths which keep track of when an edge group embeds in a vertex group as a non maximal cyclic subgroup. Therefore it seemed worthwhile recording in Theorem 10.8 that a finite graph of groups with torsion free word hyperbolic vertex groups and infinite cyclic edge groups has word hyperbolic fundamental group exactly when the graph of groups contains no complete conjugacy paths and no full non maximal paths. Moreover there are only finitely many possibilities for each of these two types of paths to occur.

2 Examples of balanced groups

The famous Baumslag-Solitar groups $BS(m, n)$ for $m, n \neq 0$ are given by the presentation $\langle a, t | ta^mt^{-1} = a^n \rangle$ and are HNN extensions where the base $\langle a \rangle$ and the associated subgroups $\langle a^m \rangle$ and $\langle a^n \rangle$ are all infinite cyclic, although note that they can also be expressed as an HNN extension with base $\langle a, b | a^n = b^m \rangle$ and stable letter t conjugating a to b ; this latter base is not infinite cyclic

unless one of $|m|, |n|$ is equal to 1. Here we will divide them up into the following categories:

If $|m| = |n|$ then we call $BS(m, n)$ **Euclidean** (after [7]), otherwise it is non-Euclidean.

If one of $|m|$ or $|n|$ is equal to 1 then $BS(m, n)$ is **soluble**, otherwise it is non-soluble.

Thus the Euclidean soluble Baumslag-Solitar groups are exactly \mathbb{Z}^2 and the Klein bottle group. Euclidean Baumslag-Solitar groups should be regarded as generally very well behaved, for instance they are linear over \mathbb{Z} and therefore over \mathbb{C} and are residually finite, they are subgroup separable (every finitely generated subgroup is an intersection of finite index subgroups, thus again they are residually finite) and we shall see shortly that they are balanced, which is a definition from [17]:

Definition 2.1 *A group G is called **balanced** if for any element x in G of infinite order we have that x^m conjugate to x^n implies that $|m| = |n|$.*

Here we will also define:

*A **balanced element** in a group G is an element x in G of infinite order such that if we have $m, n \in \mathbb{Z}$ with x^m conjugate to x^n in G then $|m| = |n|$.*

Thus a group is balanced if and only if all its elements of infinite order are balanced.

The soluble Baumslag-Solitar groups (minus the two Euclidean ones) can in turn be regarded as moderately well behaved: they are linear over \mathbb{C} , indeed they embed in $SL(2, \mathbb{C})$, thus are again residually finite, but they are not linear over \mathbb{Z} , they are not subgroup separable - indeed this fails on the infinite cyclic subgroup $\langle a \rangle$ - and they are clearly not balanced. Meanwhile the Baumslag-Solitar groups which are neither Euclidean nor soluble can be regarded as very badly behaved indeed: they are famously not residually finite and obviously not balanced, so any group having one of these as a subgroup will also fail these two conditions.

We first quote some basic properties of balanced groups, then provide a range of examples.

Lemma 2.2 ([17] Lemmas 4.13 and 4.14)

- (i) If G and H are both balanced groups then so is $G \times H$ and $G * H$.
- (ii) If G is a balanced group then so is any subgroup H of G . Conversely if G contains a finite index subgroup H which is balanced then G is balanced.

However being balanced does not hold under extensions: for instance the soluble Baumslag-Solitar group $BS(1, n)$ for $n > 1$ has a torsion free abelian normal subgroup with infinite cyclic quotient.

All word hyperbolic groups are balanced: indeed this fact is established on the way to showing that a word hyperbolic group cannot contain any Baumslag-Solitar subgroup. As for variations and generalisations on this result, it will be immediate from Lemma 5.1 later in this paper that if a group is torsion free and hyperbolic relative to a collection of balanced subgroups then it is also balanced. Also groups that act properly and cocompactly on a CAT(0) space are balanced (thus the Burger-Mozes examples are torsion free simple groups that are balanced). A torsion free word hyperbolic group is CSA (conjugate separated abelian), meaning that the centraliser of any non trivial element is abelian and malnormal: CSA groups are also clearly balanced (indeed we must have $m = n$ here). Another large class of balanced groups is the class of 3-manifold groups: interestingly this was shown in [9] well before the proof of Geometrization.

Abelian groups are obviously balanced but what about replacing abelian by nilpotent/polycyclic/soluble? Once again $BS(1, n)$ is a soluble counterexample but otherwise this holds by the following which is [17] Lemma 4.12.

Lemma 2.3 *If G is subgroup separable then it is balanced; indeed if $gx^mg^{-1} = x^n$ holds in G for x of infinite order and $|m| \neq |n|$ then one of the infinite cyclic subgroups $\langle x^m \rangle$ or $\langle x^n \rangle$ is not separable in G .*

As it is a result of Mal'cev that virtually polycyclic groups are subgroup separable, we have that they are also balanced.

Further examples can be obtained by using residual properties.

Proposition 2.4 *If G is a group which is residually (torsion free balanced), that is for all non trivial $g \in G$ we have a homomorphism onto a torsion free balanced group with g not in the kernel (so G is itself torsion free), then G is balanced.*

Proof. If $gx^mg^{-1} = x^n$ holds in G for $m \neq n$ then the commutator $[g, x]$ is non trivial, so take a homomorphism where $[g, x]$ does not vanish in the image, thus neither does g and x . As our relation still holds but we are now in a torsion free balanced group, we find that $|m| = |n|$. □

The most common families coming under Proposition 2.4 are residually free groups, or more generally residually (torsion free nilpotent) groups. In particular this is one way to see that all limit groups and all right angled Artin groups are balanced.

As $F \times \mathbb{Z}$ is balanced for F a free group, this tells us that the Euclidean Baumslag-Solitar groups are balanced because they are virtually $F \times \mathbb{Z}$. We have a variation on this:

Proposition 2.5 *A group G that is virtually (free by cyclic) is balanced.*

Proof. We can assume that G is free by cyclic by Lemma 2.2 (ii) which here will mean $G = F \rtimes \mathbb{Z}$ for F free but not necessarily finitely generated.

If $gx^mg^{-1} = x^n$ for $m \neq n$ then x must lie in the kernel of the projection to \mathbb{Z} , thus it and gxg^{-1} lie in F . But $(gxg^{-1})^m = x^n$ implies that x and gxg^{-1} are both contained in the same maximal cyclic subgroup $\langle z \rangle$ of F , so that $x = z^k$ and $gxg^{-1} = z^l$ for some k, l where $nk = ml$. But now $(gzg^{-1})^{mk} = z^{nk}$ implies that gzg^{-1} also lies in $\langle z \rangle$, so if $gzg^{-1} = z^j$ then $j = \pm 1$ (by also considering $g^{-1}zg$) and $z^{nk} = x^n = (gxg^{-1})^m = (gzg^{-1})^{mk} = z^{\pm mk}$ so $|m| = |n|$. \square

Our last set of examples involve linear groups, but not over \mathbb{C} as we know that $BS(1, n)$ lies in $GL(2, \mathbb{C})$, in fact in $GL(2, \mathbb{Q})$ and even in $SL(2, \mathbb{Q})$ if n is a square, but is not balanced for $n > 1$. In fact we consider subgroups of $GL(n, \mathbb{Z})$, which is of interest because right angled Artin groups and “cubulated” groups in the sense of Wise embed in $GL(n, \mathbb{Z})$. One might ask for a group theoretic obstruction to being linear over \mathbb{Z} (excluding those that are obstructions to being linear over \mathbb{C} such as being finitely generated but not residually finite). We only know of one, again due to Malcev, which is that every soluble subgroup must be polycyclic. Thus we might ask whether failure to be balanced is also an obstruction and indeed this is true.

Theorem 2.6 *If $gx^mg^{-1} = x^n$ holds in a group G where $|m| \neq |n|$ and x has infinite order then G is not linear over \mathbb{Z} .*

The proof is given in Section 3, although one can also establish this by using linear algebra over \mathbb{C} . Oddly in both cases linearity over \mathbb{Z} is never used other than to quote Malcev’s result.

3 The intersector

In the last section we considered balanced groups. We now go into a little more detail and look at balanced elements.

If H is a subgroup of G then we can consider the set of elements $g \in G$ such that $gHg^{-1} \cap H \neq \{e\}$. In Geometric Group Theory these have been called the intersecting conjugates of H (as in [18] p26) although the set is also known as the generalised normaliser of H in Combinatorial Group Theory, as defined in [8]. Now this set will not form a subgroup in general but a basic yet fundamental observation here is that it will if H is infinite cyclic.

Definition 3.1 *If a is an element of infinite order in a group G then we define the **intersector** $I_G(a)$ of a in G to be*

$$\{g \in G : g\langle a \rangle g^{-1} \cap \langle a \rangle \neq \{e\}\}.$$

Lemma 3.2 (i) $I_G(a)$ is a subgroup of G containing $\langle a \rangle$.

(ii) If $a^k = b^l$ for some $k, l \neq 0$ then $I_G(a) = I_G(b)$.

(iii) For any $g \in G$ we have $I_G(gag^{-1}) = gI_G(a)g^{-1}$.

Proof. (i) The identity and inverses are clearly in $I_G(a)$ so suppose $ga^i g^{-1} = a^j$ and $ha^k h^{-1} = a^l$ where all of i, j, k, l are non zero then $gha^{ik}(gh)^{-1} = a^{jl}$.
(ii) If $g \in I_G(a)$, so that $ga^i g^{-1} = a^j$ for $i, j \neq 0$, then $gb^{il} g^{-1} = ga^{ik} g^{-1} = a^{jk} = b^{jl}$ and we can now swap a and b .
(iii) Conjugation by g is an automorphism of G and the definition of $I_G(a)$ is purely group theoretic. □

Thus $I_G(a) = \langle a \rangle$ if and only if $\langle a \rangle$ is a *malnormal* subgroup of G . Indeed Lemma 3.2 (i) was already in [8], which applied it to circumstances where $\langle a \rangle$ was malnormal or close to being malnormal in G but here we are interested in the general setting. Also $I_G(a)$ is equal to the commensurator of $\langle a \rangle$ in G .

We now introduce the idea of the modular homomorphism, which can be thought of as a variation on the concept of the same name which is defined for generalised Baumslag-Solitar groups. Here we can work with an arbitrary group but we only obtain a “local” version.

Definition 3.3 *Given a group G and an element $a \in G$ of infinite order, the **modular homomorphism** Δ_a^G of a in G (or Δ_a when the group is clear) is the map from the intersector $I_G(a)$ to the multiplicative non zero rational numbers \mathbb{Q}^* defined as follows: if $g \in I_G(a)$ so that $ga^m g^{-1} = a^n$ for some $m, n \in \mathbb{Z} \setminus \{0\}$ then we set $\Delta_a^G(g) = m/n \in \mathbb{Q}^*$.*

It is not hard to see that Δ_a^G is well defined (because a has infinite order) and is a homomorphism (this homomorphism was noted in [9] although with domain restricted to $\langle g, a \rangle$ for a given $g \in I_G(a)$). Note also that if g has finite order then $\Delta_a^G(g) = \pm 1$.

Definition 3.4 *The modulus of $a \in G$ is the image of Δ_a^G in \mathbb{Q}^* and a is called a **unimodular** element of G if its modulus is contained in $\{\pm 1\}$.*

Note that if we have two elements $a, b \in G$ of infinite order with a non trivial power of a conjugate to a power of b then a and b have the same modulus by repeated use of Lemma 3.2 (ii) and (iii). Also we see that a group is balanced if and only if all its elements of infinite order are unimodular.

In the previous section we gave many examples of balanced groups, but the only unbalanced groups mentioned were the non Euclidean Baumslag-Solitar groups. Of course any group with one of these as a subgroup would also be unbalanced, but we have not yet seen any unbalanced groups which do not contain any Baumslag-Solitar group.

Example Let r, s be non zero coprime integers where $|r|$ and $|s|$ are not both 1 and consider the ring $\mathbb{Z}[1/rs]$ but considered as a torsion free abelian group A , here written additively, which is locally cyclic as it is a subgroup of \mathbb{Q} . Let us now form the semidirect product $G_{r,s} = G = A \rtimes \mathbb{Z}$ where the generator t of \mathbb{Z} acts on $a \in A$ by conjugating a to $(s/r)a$, which is a metabelian group. Now any element of $G_{r,s}$ can be written in the form $x = (a, t^k)$, with $G_{r,s}$ generated by $(1, e)$ and $(0, t)$, and if $k \neq 0$ then x is a balanced element (for instance we can project into \mathbb{Z}), thus we see that the set of unbalanced elements in G is exactly $A \setminus \{0\}$, with the modulus of each element equal to $\{(r/s)^k : k \in \mathbb{Z}\}$. We then have that if neither $|r|$ nor $|s|$ is 1 then $G = G_{r,s}$ does not contain any Baumslag-Solitar subgroup: first if G contains $BS(m, n)$ for $|m| \neq |n|$ then neither $|m|$ nor $|n|$ can equal 1 because the only integer contained in the modulus of any $a \in A$ is 1. Also A does not contain \mathbb{Z}^2 so if G contained such a subgroup $H \cong \mathbb{Z}^2$ then H would have non trivial intersection with A but would also have non trivial image under projection to the \mathbb{Z} factor, so would contain an element of the form (a, t^k) for $k \neq 0$ which does not commute with anything in $A \setminus \{0\}$. Thus G can only contain non soluble Baumslag Solitar groups but it is soluble.

If N is the normal closure of a in $BS(r, s) = \langle a, t | ta^r t^{-1} = a^s \rangle$ and N' the commutator subgroup of N then it is well known that $G_{r,s}$ is isomorphic to $BS(r, s)/N'$, with a mapping to $(1, e)$ and t to $(0, 1)$. Here if one of $|r|$ or

$|s| = 1$ then N' is trivial. Other interesting properties of this correspondence between $BS(r, s)$ and $G_{r,s}$ when neither $|r|$ nor $|s|$ equal 1 are that $G_{r,s}$ is not finitely presented ([2]) and that the finite residual R of $BS(r, s)$ (the intersection of all finite index subgroups) is equal to N' [15]. These observations can be used to give fairly general circumstances under which a group which is not balanced will contain a non Euclidean Baumslag Solitar subgroup.

Lemma 3.5 *If we have a surjective homomorphism of $G_{r,s}$ in which the image of $(1, e)$ has infinite order then this is an isomorphism.*

Proof. If (a, e) is in the kernel for $a \in A \setminus \{0\}$ then so is (n, e) for some non zero integer n . Otherwise we have some element (\cdot, t^k) in the kernel for $k \neq 0$, but as this element conjugates (x, e) into $((s/r)^k x, e)$ and $(s/r)^k$ is never 1 here, we again have $a \in A \setminus \{0\}$ such that (a, e) is in the kernel too, thus so is some power of $(1, e)$. □

Proposition 3.6 *Let G be an unbalanced group.*

- (i) *If some modulus contains an integer not equal to ± 1 then G must contain a Baumslag-Solitar subgroup $BS(1, n)$ for $|n| > 1$.*
- (ii) *If G is residually finite then G must contain a subgroup of the form $G_{r,s}$.*
- (iii) *If G is residually finite and coherent (meaning that every finitely generated subgroup is finitely presented) then G must contain a Baumslag-Solitar subgroup $BS(1, n)$ for $|n| > 1$.*

Proof. If (i) holds then we have infinite order elements $g, a \in G$ with $ga^m g^{-1} = a^n$ where $m = ln$ for $|l| > 1$. Thus on replacing a with a^n we have $g^{-1}ag = a^l$ so that $\langle a, g \rangle$ is a homomorphic image of $BS(1, l)$. But as $G_{1,l} = BS(1, l)$ and a has infinite order we have by Lemma 3.5 that $\langle a, g \rangle$ is isomorphic to $BS(1, l)$.

If now G is residually finite but unbalanced, so we have $ga^r g^{-1} = a^s$ for $|r| \neq |s|$ and $|r|, |s|$ coprime without loss of generality then again $\langle a, g \rangle$ is a homomorphic image of $BS(r, s)$. But it is a residually finite image so the homomorphism must factor through $BS(r, s)/R = G_{r,s}$. Then we can again apply Lemma 3.5.

If however G is also coherent then $G_{r,s}$ cannot be a subgroup of G unless one of $|r|$ or $|s|$ is 1 (so say $r = 1$), in which case being unbalanced implies that we have infinite order elements $a, g \in G$ with $gag^{-1} = a^s$ for $|s| > 1$,

thus s is in the modulus of the element a of G . In particular in a residually finite coherent group, every modulus consists only of integers and their reciprocals.

□

Another application to unbalanced groups is that of CT (commutative transitive) groups, such as torsion free subgroups of $SL(2, \mathbb{C})$. These are generalisations of CSA groups, with the latter always balanced so we might expect CT groups to be as well. However the fact that $G_{r,s}$ is a subgroup of $SL(2, \mathbb{C})$ via the embedding

$$(1, e) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (0, t) \mapsto \begin{pmatrix} \sqrt{\frac{s}{r}} & 0 \\ 0 & \sqrt{\frac{r}{s}} \end{pmatrix}$$

(which is injective by Lemma 3.5 and so is CT) provides immediate counterexamples - though in fact essentially the only counterexamples.

Proposition 3.7 *If G is a CT group that is not balanced then it contains an isomorphic copy of $G_{r,s}$ for $|r| \neq |s|$.*

Proof. On again taking $ga^rg^{-1} = a^s$ for g, a elements of infinite order and r, s coprime but $|r| \neq |s|$, we have that a commutes with ga^rg^{-1} which commutes with gag^{-1} , so gag^{-1} commutes with a , whereupon g^2ag^{-2} commutes with gag^{-1} and thus with a , and so on. Thus the normal closure of a in $\langle a, g \rangle$ is abelian, so that $\langle a, g \rangle$ is a homomorphic image of $BS(r, s)/N' = G_{r,s}$ and so is equal to $G_{r,s}$ by Lemma 3.5.

□

Finally we can provide the proof of our result left over from the last section.

Theorem 3.8 *A subgroup of $GL(n, \mathbb{Z})$ is balanced.*

Proof. On taking the usual $\langle a, g \rangle$ with $ga^rg^{-1} = a^s$ for r, s coprime and $|r| \neq |s|$, we have that $\langle a, g \rangle$ is a finitely generated linear group, thus residually finite so Proposition 3.6 (ii) applies. But $G_{r,s}$ is then a soluble subgroup of $GL(n, \mathbb{Z})$ so must be polycyclic which is a contradiction.

□

4 Acylindrical hyperbolicity of graphs of groups with infinite cyclic edge groups

We now turn to a topic which, on the face of it, seems to have little to do with balanced groups. In [14] a subgroup H of a group G is called *weakly malnormal* if there is $g \in G$ such that $gHg^{-1} \cap H$ is finite, and *s-normal* otherwise. Thus if we take $H = \langle a \rangle$ to be infinite cyclic, we see that $\langle a \rangle$ being *s-normal*/weakly malnormal is equivalent to $I_G(a)$ being equal/not equal to G . In that paper this concept was introduced in the context of acylindrical hyperbolicity, with a group being acylindrically hyperbolic implying that it is SQ -universal and in particular is not a simple group. The paper gave conditions under which the fundamental group of a graph of groups is acylindrically hyperbolic and then provided related applications. Here we will stick to the case where all edge groups are infinite cyclic, whereupon we can use intersectorors to determine exactly when a finite graph of groups with infinite cyclic edge groups is acylindrically hyperbolic.

The relevance of *s-normal* subgroups in this setting is twofold: first if a group G is acylindrically hyperbolic then any *s-normal* subgroup of G is itself acylindrically hyperbolic: in particular it cannot be a cyclic subgroup. Also in Section 4 of [14] we have sufficient conditions for acylindrical hyperbolicity which we now describe: given a graph of groups $G(\Gamma)$ with connected graph Γ and fundamental group G , an edge e is called *good* if both edge inclusions into the vertex groups at either end of e give rise to proper subgroups, otherwise it is *bad*. A *reducible* edge is a bad edge which is not a self loop. Given a finite graph of groups, we can contract the reducible edges one by one until none are left. This process does not affect the fundamental group G and the new vertex groups will form a subset of the original vertex groups. It could be that we are left with a single vertex and no edges, in which case we say that the graph of groups $G(\Gamma)$ was trivial with G equal to the remaining vertex group. We then have:

Theorem 4.1 ([14] Theorem 4.17) *Suppose that $G(\Gamma)$ is a finite reduced graph of groups which is non trivial and which is not just a single vertex with a single bad edge. If there are edges e, f of Γ (not necessarily distinct) with edge groups G_e, G_f and an element $g \in G$ such that $G_f \cap gG_eg^{-1}$ is finite then G is either virtually cyclic or acylindrically hyperbolic.*

Consequently we might hope that if we have a graph of groups with infinite

cyclic edge groups then the existence of s -normal cyclic subgroups characterises whether the fundamental group of this graph of groups is acylindrically hyperbolic, which indeed turns out to be the case. In fact we can characterise acylindrical hyperbolicity directly from the graph of groups.

Theorem 4.2 *Suppose that G is the fundamental group of a non trivial finite reduced graph of groups $G(\Gamma)$ where all edge groups are infinite cyclic, but with no further restriction on the vertex groups. Then G is acylindrically hyperbolic unless both of the following two conditions hold:*

- (i) *At each vertex $v \in \Gamma$ with vertex group G_v , the intersection of all the inclusions of the edge groups incident at v is a non trivial subgroup of G_v .*
- (ii) *If (i) holds, so that at each vertex v we can let $g_v \neq e$ be a generator of this intersection, then we have $I_{G_v}(g_v) = G_v$.*

In this case $\langle g_v \rangle$ is s -normal in G for any $v \in \Gamma$ and so G is not acylindrically hyperbolic.

Proof. First let us assume that (i) fails. As a finite intersection of infinite cyclic subgroups is trivial if and only if all pairwise intersections are trivial, we will have a vertex $v \in \Gamma$ with the images in the vertex group G_v under inclusion of two edge groups being $\langle a \rangle$ and $\langle b \rangle$ say where $\langle a \rangle \cap \langle b \rangle$ is trivial. Consequently G_v and hence G will not be virtually cyclic, which also means we do not have a single vertex with a single bad edge as the edge subgroups are infinite cyclic, thus we can immediately apply Theorem 4.1.

Now we suppose that (i) holds but at some vertex v we have the generator $g_v \neq e$ of the intersection of the edge group inclusions has $I_{G_v}(g_v) \neq G_v$. Then $I_{G_v}(g_v) = I_{G_v}(a)$ for a a generator of one of the edge group inclusions into G_v . Thus there is an element $x \in G_v \setminus I_{G_v}(a)$, meaning that $\langle a \rangle \cap x\langle a \rangle x^{-1}$ is trivial and G_v is not virtually cyclic. So we again apply Theorem 4.1.

Conversely if (i) and (ii) both hold then there exists an element $g \in G$ which is a power of g_v for every $v \in \Gamma$. To see this, first take a maximal tree T in Γ and form the group G_T from this tree using amalgamated free products. We argue by induction on the number of edges: suppose that $T = T_0 \cup \{e_l\}$ where e_l is a leaf edge and T_0 has e_l removed, with G_{T_0} the fundamental group of this graph of groups. Suppose that $g_0 \in G_{T_0}$ is a power of every g_v for $v \in T_0$. Then on adding the edge e_l with new vertex v_1 to T_0 at the vertex $v_0 \in T_0$, the edge group of e_l provides the inclusion $\langle a \rangle$ into G_{v_0} and G_{v_1} . Now $\langle g_{v_0} \rangle \leq \langle a \rangle$ with $g_{v_0} \neq e$ by condition (i) and we are supposing that g_0 is a power of g_{v_0} , hence a power of a . But at the vertex v_1 we have that g_{v_1} is also a power of a , telling us that an appropriate power g of g_0

is also a power of g_{v_1} , and thus of every g_v for $v \in T$. This then confirms our claim for the group G_T , but this embeds in G on introducing the stable letters for the edges in $\Gamma \setminus T$ and forming the HNN extensions. Moreover G is generated by its vertex groups G_v and those stable letters, so that we certainly have all G_v in $I_G(g)$ by condition (ii) and Lemma 3.2 (ii). Finally each edge in $\Gamma \setminus T$ provides a stable letter t and edge group inclusions $\langle a \rangle$ into some G_v and $\langle b \rangle$ into some G_w (possibly $v = w$) where $tat^{-1} = b$. But our element g is a power of both a and b , so that t is in $I_G(g)$ as well and hence $I_G(g) = G$. Thus as g is a power of any g_v , we have that $I_G(g_v) = G$ too and hence G is not acylindrically hyperbolic. \square

We can now use this to show that a finitely generated simple group cannot split over \mathbb{Z} . It is known that a fundamental group of a non trivial finite graph of groups with all edge groups finite cannot be simple by [12].

Theorem 4.3 *If $G(\Gamma)$ is a non trivial graph of groups with finitely generated fundamental group G and where all edge groups are infinite cyclic then G is either:*

- (i) *acylindrically hyperbolic or*
- (ii) *has a homomorphism onto \mathbb{Z} or*
- (iii) *has an infinite cyclic normal subgroup.*

In particular G is never simple.

Proof. As G is finitely generated, we can assume that Γ is a finite graph, and then as above we can assume that $G(\Gamma)$ is reduced. It is well known that if Γ is not a tree then G surjects to \mathbb{Z} as $\pi_1(\Gamma)$ is non trivial. We now apply Theorem 4.2 to obtain acylindrical hyperbolicity of G unless both (i) and (ii) hold, in which case we have our infinite order element $g \in G$ which lies in every vertex group G_v and such that $I_{G_v}(g) = G_v$, but $I_G(g)$ is a subgroup of G containing all G_v and thus is equal to G as Γ is a tree. It is here that the ideas of balanced groups and elements reenter: first suppose that g is unimodular in every vertex group. Then $\Delta_g^{G_v}(G_v)$ is contained in the subgroup ± 1 of \mathbb{Q}^* and therefore so is $\Delta_g^G(G)$. Thus for all $x \in I_G(g) = G$ we can find an integer $k > 0$ such that $xg^kx^{-1} = g^{\pm k}$. Here k depends on x but as we have a finite generating set for G we can find a common power P that works for all of this set and hence for all of G , thus $\langle g^P \rangle$ is normal in G .

Now suppose that we have $v \in \Gamma$ such that $\Delta_g^{G_v}$ is not contained in ± 1 . We borrow a trick from [9], which is that $|\Delta_g^{G_v}|$ provides a homomorphism from G_v to the positive rationals which is non trivial, thus the image is an infinite torsion free abelian group and g is in the kernel. However this means that g_v is too and hence also g_e for g_e the generator of any edge group with edge e incident at v . Thus on sending all other vertex groups to the identity, we extend the domain of this homomorphism to all of G , which being finitely generated means that the infinite torsion free abelian image of G must be \mathbb{Z}^n .

□

5 Absence of relative hyperbolicity

For the remainder of the paper we will consider only torsion free groups, so as to allow for clean statements that do not require consideration of many cases in the corresponding proofs. Henceforth element will mean a group element of infinite order and power will mean a non zero power, thus for instance saying that elements x, y in G have conjugate powers is a shorthand for saying that for all $i, j > 0$ we have $x^i, y^j \neq e$ but there exists $g \in G$ and non zero integers m, n such that $gx^m g^{-1} = y^n$.

We now examine whether the fundamental group of a graph of groups is relatively hyperbolic with respect to a collection of proper subgroups. As this would imply acylindrical hyperbolicity anyway (at least if the group is not virtually cyclic), our emphasis will be on finding conditions that ensure that the groups considered in the previous section are not hyperbolic with respect to any collection of proper subgroups. For this we adopt the method from [5] Section 4, which itself borrows from [1]. We first summarise the facts we need about relatively hyperbolic groups, all of which come from [16]. We suppose that G is hyperbolic relative to a collection of proper subgroups H_1, \dots, H_l , the peripheral subgroups, and we say that $g \in G$ is hyperbolic if g is not conjugate into a peripheral subgroup. We also assume here for these statements that G is torsion free. We then have:

- ([16] Theorem 4.19) If $g \in G$ is hyperbolic then the centraliser $C_G(g)$ is strongly relatively quasiconvex in G . (Here strongly means that its intersection with any conjugate of a peripheral subgroup is finite, as in [16] Definition 4.11.)

- ([16] Theorem 4.16) A strongly relatively quasiconvex subgroup of G is word hyperbolic.
- ([16] Corollary 4.21) If g is a hyperbolic element in G and we have $t \in G$ with $tg^kt^{-1} = g^l$ then $|k| = |l|$.
- ([16] Theorem 1.4) Any H_i is malnormal, so that if there is $g \in G$ with $H_i \cap gH_i g^{-1}$ non trivial then $g \in H_i$. Moreover if there is $g \in G$ with $H_i \cap gH_j g^{-1}$ non trivial then $i = j$.

We know that in a torsion free word hyperbolic group G (where we can just take the single peripheral subgroup $\{e\}$) the centraliser $C_G(g)$ of any non identity element g is a maximal cyclic subgroup of G , but we can see that this also holds for the intersector. Indeed we have:

Lemma 5.1 *For any non identity g in a group G which is torsion free and hyperbolic relative to a collection of proper subgroups, either g is hyperbolic in which case $I_G(g)$ is a maximal cyclic subgroup of G or $I_G(g)$ is conjugate into a peripheral subgroup.*

Proof. If g is hyperbolic then g^k is also, as if $g^k \in \gamma P \gamma^{-1}$ for P a peripheral subgroup and $\gamma \in G$ then $g^k \in g(\gamma P \gamma^{-1})g^{-1} \cap \gamma P \gamma^{-1}$, so $g \in \gamma P \gamma^{-1}$ by malnormality. But the first two points above say that $C_G(g)$ is a word hyperbolic group which here will also be torsion free with an infinite centre, so it must be infinite cyclic (and maximal in G as it is a centraliser), say $C_G(g) = \langle h \rangle$. Now let us take $t \in I_G(g)$, so that we have $tg^kt^{-1} = g^l$ for $|k| = |l|$, by the above. If $l = k$ then we see that t commutes with g^k , which has the same centraliser as g (because g^k is hyperbolic with $C_G(g) \leq C_G(g^k)$ and these are maximal cyclic subgroups), so that t is in $C_G(g)$ too. If $l = -k$ then t^2 commutes with g^k so say $t^2 = h^i$ and $g = h^j$, whereupon $(tg^kt^{-1})^i = g^{-ik}$ implies that

$$tt^{2jk}t^{-1} = th^{ijk}t^{-1} = g^{-ik} = h^{-ijk} = t^{-2jk}$$

so that t is a torsion element which is a contradiction.

Now suppose that g lies in $\gamma P \gamma^{-1}$ for P a peripheral subgroup and $\gamma \in G$. If $t \in I_G(g)$ so that $tg^it^{-1} = g^j$ then we have $g^j \in t(\gamma P \gamma^{-1})t^{-1} \cap \gamma P \gamma^{-1}$ so by malnormality as before we obtain $t \in \gamma P \gamma^{-1}$.

□

We can now prove under very general circumstances that graphs of groups are not relatively hyperbolic. Of course the condition in the theorem below on non trivial edge groups is needed, otherwise we can obtain free products which will be relatively hyperbolic with respect to the factors.

Theorem 5.2 *Let $G(\Gamma)$ be a finite graph of groups where each vertex group is non trivial and torsion free but with no further restriction on the edge groups other than they are all non trivial. Suppose that G is relatively hyperbolic with respect to the collection of subgroups H_1, \dots, H_l . Suppose further that for all vertices $v \in \Gamma$, there is a peripheral subgroup $H_{i(v)}$ such that the vertex group G_v is conjugate in G into $H_{i(v)}$. Then some peripheral subgroup is equal to G , so that G is not hyperbolic relative to any collection of proper subgroups.*

Proof. On picking a maximal tree in Γ and any vertex v , there is $\gamma \in G$ and a peripheral subgroup P such that $G_v \subseteq \gamma P \gamma^{-1}$. But for any w adjacent to v in T we similarly have $\delta \in G$ and a peripheral subgroup Q such that $G_w \subseteq \delta Q \delta^{-1}$. But $G_v \cap G_w$ is non trivial as it contains this edge group so $\gamma P \gamma^{-1} = \delta Q \delta^{-1}$ by the malnormal property for peripheral subgroups mentioned above. This means that $P = Q$ and $\delta^{-1} \gamma \in P$ so G_v and G_w are in $\gamma P \gamma^{-1}$. We now continue until we find that G_v is in the same conjugate $\gamma P \gamma^{-1}$ of the same peripheral subgroup P for all $v \in \Gamma$.

We now add the stable letters t_i : as each t_i conjugates a non trivial subgroup of G_v to one of G_w for some $v, w \in \Gamma$, we see that $t_i(\gamma P \gamma^{-1})t_i^{-1} \cap \gamma P \gamma^{-1}$ is non trivial, so t_i and hence the whole group G is in $\gamma P \gamma^{-1}$, thus $G = P$. \square

We now need to give conditions on the vertex subgroups of a graph of groups $G(\Gamma)$ in order to ensure that the conditions of Theorem 5.2 apply. By Lemma 5.1 if each vertex group G_v has an element whose intersector is all of G_v and G_v is not infinite cyclic then we can apply Theorem 5.2 to conclude that G is not relatively hyperbolic. However we can repeat this idea by taking bigger and bigger subgroups of G_v with the same properties, thus giving rise to the next definition.

Definition 5.3 *If H is any non trivial subgroup of a group G then we define the **full intersecting conjugate** $F_G(H) = F_G^1(H)$ of $H = F_G^0(H)$ in G to be the subgroup*

$$\langle g \in G : gHg^{-1} \cap H \neq \{e\} \rangle$$

of G which contains H . We then inductively define

$$F_G^{n+1}(H) = F_G(F_G^n(H)) \text{ and } \text{Mal}_G(H) = \cup_{n=1}^{\infty} F_G^n(H).$$

(Note for an element $a \in G$ of infinite order we have $F_G^1(\langle a \rangle) = F_G(\langle a \rangle) = I_G(a)$.) This is an ascending union of subgroups so also a subgroup and has the following properties:

Lemma 5.4 (i) If $S \leq H$ and $H \leq G$ then $Mal_G(S) \leq Mal_G(H)$ and $Mal_H(S) \leq Mal_G(S)$.

(ii) $Mal_G(H)$ is malnormal in G .

(iii) If M is a malnormal subgroup of G containing H then M contains $Mal_G(H)$.

Proof. Part (i) follows directly from the definition so first say that we have $g \in Mal_G(H)$ such that $gMal_G(H)g^{-1} \cap Mal_G(H) \neq \{e\}$, so we have $m_1, m_2 \in Mal_G(H) \setminus \{e\}$ with $gm_1g^{-1} = m_2$. Then we have $N \in \mathbb{N}$ with $m_1, m_2 \in F_G^N(H)$, thus $gF_G^N(H)g^{-1} \cap F_G^N(H) \neq \{e\}$ and therefore $g \in F_G(F_G^N(H)) \subseteq Mal_G(H)$.

Now suppose inductively that $F_G^N(H) \subseteq M$. If $gF_G^N(H)g^{-1} \cap F_G^N(H) \neq \{e\}$ then $gMg^{-1} \cap M \neq \{e\}$ giving $g \in M$, so all generators of $F_G^{N+1}(H)$ are in the subgroup M thus $F_G^{N+1}(H)$ is also and $Mal_G(H)$ is the union of these. \square

Thus one could perhaps call $Mal_G(H)$ the malnormal closure of H in G .

Corollary 5.5 Let $G(\Gamma)$ be a finite graph of groups where each vertex group is non trivial and torsion free but with no further restriction on the edge groups other than they are all non trivial. Suppose that for all vertices $v \in \Gamma$ we have $g_v \in G_v$ such that $Mal_{G_v}(\langle g_v \rangle) = G_v$. Then G is not hyperbolic relative to any collection of proper subgroups.

Proof. First suppose that no vertex groups are copies of \mathbb{Z} . Then we can assume that $F_G^1(\langle g_v \rangle)$ is non cyclic, because if we have an element g in a group G with $I_G(g) = F_G^1(\langle g \rangle) = \langle c \rangle$ then $F_G^2(\langle g \rangle) = F_G^1(\langle c \rangle) = I_G(c)$, whereas $g = c^i$ means that $I_G(g) = I_G(c)$ and thus our ascending union $Mal_G(\langle g \rangle)$ terminates in the cyclic group $\langle c \rangle$, but here $Mal_G(\langle g_v \rangle)$ contains $Mal_{G_v}(\langle g_v \rangle) = G_v$ by Lemma 5.4 (i). Thus by Lemma 5.1 we see that we have a peripheral subgroup P and an element $\gamma \in G$ such that $I_G(g_v) = F_G^1(\langle g_v \rangle) \subseteq \gamma P \gamma^{-1}$. But by malnormality of $\gamma P \gamma^{-1}$ in G we have by Lemma 5.4 (iii) that $\gamma P \gamma^{-1}$ contains $Mal_G(\langle g_v \rangle)$ which itself contains $Mal_{G_v}(\langle g_v \rangle) = G_v$, so now we can apply Theorem 5.2.

Now say that some vertex groups are copies of \mathbb{Z} . We begin as before by taking a vertex group $G_{v_0} \not\cong \mathbb{Z}$ (if none exist then we have a generalised

Baumslag-Solitar group as in [10] which has an s -normal infinite cyclic subgroup and so cannot even be acylindrically hyperbolic) and proceed similarly, so that G_{v_0} is in $\gamma P \gamma^{-1}$ along with G_v for all other vertices encountered so far. If at any stage we now have $G_w = \langle z \rangle \cong \mathbb{Z}$, where w is adjacent to v in T and we already have $G_v \subseteq \gamma P \gamma^{-1}$ then we find that $z^i \in G_v$, where $\langle z^i \rangle$ is the edge group inclusion into G_w , so $z^i \in \gamma P \gamma^{-1}$ but then $z^i \in z(\gamma P \gamma^{-1})z^{-1} \cap \gamma P \gamma^{-1}$ means that z , and hence G_w , is in $\gamma P \gamma^{-1}$ too. \square

Examples

If a torsion free group S is soluble then let us take an element s in the final non trivial term $S^{(n)}$ of the derived series. Clearly $s \in S^{(n)} \leq F_S^1(\langle s \rangle)$ because $S^{(n)}$ is abelian, but then $S^{(i+1)}$ being normal in $S^{(i)}$ means that all $S^{(i)}$ and thus S are in $Mal_S(\langle s \rangle)$ too. Now suppose we have a torsion free group H with an infinite soluble normal subgroup S as above. Then $S \leq F_H^{n+1}(\langle s \rangle)$ implies that $H \leq F_H^{n+2}(\langle s \rangle)$. Let us further suppose that the torsion free group G has a finite index subgroup H which possesses an infinite soluble normal subgroup S then, with S and s as before, we obtain $H \leq F_G^{n+2}(\langle s \rangle)$ and so $G \leq F_G^{n+3}(\langle s \rangle) \leq Mal_G(\langle s \rangle)$. Thus we have

Corollary 5.6 *Suppose that $G(\Gamma)$ is a finite graph of groups where each vertex group is torsion free and contains a finite index subgroup which itself has an infinite soluble normal subgroup, and where the edge groups are all non trivial. Then G is not hyperbolic relative to any collection of proper subgroups.*

As an example from 3-manifolds, a compact orientable irreducible 3-manifold is a graph manifold if all components in its JSJ decomposition are Siefert fibred spaces. In terms of the fundamental group, we can describe this as a graph of groups where each edge group is non trivial and where each vertex group has a finite index subgroup which in turn has an infinite cyclic normal subgroup. Thus we see that the fundamental groups of graph manifolds are never relatively hyperbolic with respect to any collection of proper subgroups, as opposed to when there are hyperbolic pieces in the decomposition. In this case it is well known that the fundamental group is hyperbolic relative to the maximal graph manifold pieces, including \mathbb{Z}^2 subgroups for tori bounding hyperbolic pieces on both sides.

6 Balanced HNN extensions and amalgamated free products

Although the property of being torsion free is preserved under HNN extensions and amalgamated free products, being torsion free and balanced is not. For HNN extensions this is obvious even with edge groups that are infinite cyclic. For amalgamated free products, we have examples such as A is the free group on a, b and X is the free group on x, y but we amalgamate the rank 2 free subgroup $C = \langle a^3, bab^{-1} \rangle$ of A with the isomorphic subgroup $\langle x^2, yxy^{-1} \rangle$ of X via $a^3 = x^2, bab^{-1} = yxy^{-1}$ to form $A *_C X$ in which $(y^{-1}b)a^2(y^{-1}b)^{-1} = a^3$ holds but a is not trivial. However in this section we will examine how intersectorors change when forming amalgamated free products or HNN extensions over an infinite cyclic edge group, as this will provide necessary and sufficient conditions as to when the property of being balanced is preserved under these constructions. These will then be applied to the more general graph of groups construction in the next two sections. First we consider amalgamated free products $G = A *_C B$. If $g \in G$ is not in C then we can express g as $g_r \dots g_2 g_1$ for length $r \geq 1$ and g_1, g_2, \dots, g_r coming alternately from $A \setminus C$ and $B \setminus C$ (not uniquely though), which we will refer to as a reduced form for g . Conversely an element of this form cannot equal the identity or lie in C ; indeed if $r \geq 2$ then it cannot even lie in $A \cup B$.

Theorem 6.1 *If A and B are balanced torsion free groups and $G = A *_C B$ for infinite cyclic C then G is also balanced.*

Proof. Suppose otherwise, so that there is $x \in G$ with a power x^m conjugate in G to x^n for $|m| \neq |n|$. On considering the action of G on the associated Bass-Serre tree, we see that all hyperbolic elements are balanced (as they have non zero translation lengths), so x must be conjugate in G into either A or B but conjugates of balanced elements are balanced. Thus without loss of generality $x = a \in A$ and $ga^m g^{-1} = a^n$ for some $g \in G$, whereupon we must have $g \notin A$ because a is balanced in A by assumption.

We first suppose that no power of a is conjugate in A into C . On expressing $g = g_r \dots g_1$ in reduced form for $r \geq 1$, we have that $a^m \in A \setminus C$, so if $g_1 \in B \setminus C$ then $ga^m g^{-1}$ is in reduced form when written as $g_r \dots g_1 a^m g_1^{-1} \dots g_r^{-1}$ and so is not even in A , let alone equal to a power of a . If however $g_1 \in A \setminus C$, whereupon we would have $r \geq 2$ as $g \notin A$, then also $g_1 a^m g_1^{-1} \in A \setminus C$ by

our assumption, so now g is in reduced form of length at least three when written as $g_r \dots g_2(g_1 a^m g_1^{-1})g_2^{-1} \dots g_r^{-1}$ and hence is not in A .

Now suppose that $C = \langle c \rangle$ and some power a^i say of a is conjugate in A into C , so by replacing a and a^i with the relevant conjugates in A we can assume a^i is equal to some power c^j of c . Establishing that c is unimodular in G also implies that c^j and hence a^i and a are all unimodular in G too. We clearly have $\langle I_A(c), I_B(c) \rangle \leq I_G(c)$ as $I_G(c) \cap H = I_H(c)$ for any subgroup H of G containing c , so we show containment the other way: given $g \in G \setminus (A \cup B)$ with $g \in I_G(c)$, so that $gc^k g^{-1} = c^l$ for some non zero k, l , we again write $g = g_r \dots g_1$ in reduced form of length $r \geq 2$. But regardless of whether $g_1 \in A \setminus C$ or $B \setminus C$, we have two cases: case 1 is that $g_1 c^k g_1^{-1}$ is in $A \setminus C$ or $B \setminus C$ and thus $gc^k g^{-1}$ is in reduced form of length at least 3 so is not in $A \cup B$. The other case is when $g_1 c^k g_1^{-1}$ is in C but then g_1 is in $I_A(c)$ or $I_B(c)$ and either way we would find that $g_1 c^k g_1^{-1} = c^{\pm k}$ as A and B are balanced. By continuing in this way with g_2, \dots, g_r , we find that either $gc^k g^{-1}$ terminates in a reduced form and so $g \notin I_G(c)$, or $gc^k g^{-1} = c^{\pm k}$ so that $|k| = |l|$ and all of g_1, \dots, g_r were in $I_A(c) \cup I_B(c)$.

□

Corollary 6.2 *If G is the fundamental group of a finite graph of torsion free, balanced groups with all edge groups infinite cyclic and the graph is a tree then G is also balanced.*

Proof. Build G up by repeated amalgamations and use Theorem 6.1 at each stage.

□

Now we come to HNN extensions, whereupon it is clear that we can create non balanced groups, for instance Baumslag-Solitar groups, from balanced ones. We suppose that G is an HNN extension of the base group H and associated isomorphic subgroups A, B of H , with $tAt^{-1} = B$. Again we need the concept of a reduced form in that any element $g \in G \setminus H$ can be expressed as $h_r t^{\epsilon_r} \dots h_1 t^{\epsilon_1} h_0$ of length $r \geq 1$, $h_i \in H$, $\epsilon_i \in \{\pm 1\}$ and no pinch (an appearance of $th_j t^{-1}$ for $h_j \in A$ or $t^{-1}h_j t$ for $h_j \in B$) occurs, and conversely an element in such form does not lie in H . This allows us to give sufficient conditions under which the HNN extension is also balanced.

Proposition 6.3 *If G is an HNN extension of the balanced, torsion free group H with stable letter t and infinite cyclic associated subgroups $A = \langle a \rangle$ and $B = \langle b \rangle$ of H so that $tat^{-1} = b$ then:*

(i) *If $h \in H$ but no power of h is conjugate in H into $A \cup B$ then h is still unimodular in G .*

(ii) *If no conjugate in H of B intersects A non trivially then G is also a balanced group.*

Proof. On being given $h \in H$, suppose that there is $g \in G \setminus H$ with $gh^i g^{-1} = h^j$ and $g = h_r t^{\epsilon_r} \dots h_1 t^{\epsilon_1} h_0$ in reduced form. Then $gh^i g^{-1}$ is also in reduced form and hence not in H unless $\epsilon_1 = +1$ and $h_0 h^i h_0^{-1} \in A$ or $\epsilon_1 = -1$ and $h_0 h^i h_0^{-1} \in B$. But neither of these occur in (i) so $I_H(h) = I_G(h)$ and h is also unimodular in G .

For (ii) we again note that by using conjugacy and the Bass-Serre tree, we need only check the unimodularity of elements in H , so by (i) we can now assume that without loss of generality some power of h lies in A but that no power of h is conjugate in H into B . Thus if $gh^i g^{-1}$ is to be an element of H in the above then we can only have $\epsilon_1 = +1$ and $h_0 h^i h_0^{-1} \in A$, say a^k so that

$$gh^i g^{-1} = h_r t^{\epsilon_r} \dots t^{\epsilon_2} h_1 b^k h_1^{-1} t^{-\epsilon_2} \dots t^{-\epsilon_r} h_r^{-1}$$

because $tat^{-1} = b$. But $h_1 b^k h_1^{-1}$ cannot be in A so again we are reduced if $\epsilon_2 = +1$, or if $\epsilon_2 = -1$ and $h_1 b^k h_1^{-1}$ is not in B . But if $h_1 b^k h_1^{-1} = b^l$ then $|k| = |l|$ because b is unimodular in H . By continuing in this way, we see that either $gh^i g^{-1}$ terminates in a reduced word not lying in H , or we merely pass through $b^{\pm k}$ or $a^{\pm k}$ as we evaluate $gh^i g^{-1}$ from the middle outwards. But if we end up at the last step with $h^j = gh^i g^{-1} = h_r b^{\pm k} h_r^{-1}$ then h^j would be conjugate in H into B . Thus here we can only end up with

$$h^j = gh^i g^{-1} = h_r a^{\pm k} h_r^{-1} = h_r h_0 h^{\pm i} h_0^{-1} h_r^{-1}$$

so $|i| = |j|$ as h is unimodular in H . □

We can now give the exact condition on when such an HNN extension would not be balanced.

Theorem 6.4 *Let G be an HNN extension of the balanced, torsion free group H with stable letter t and infinite cyclic associated subgroups $A = \langle a \rangle$ and*

$B = \langle b \rangle$ of H where $tat^{-1} = b$. Suppose further that there is $h \in H$ conjugating a power of a to a power of b , so that $ha^i h^{-1} = b^j$. Then G is balanced if and only if $|i| = |j|$.

Proof. We first replace t with the alternative stable letter $s = h^{-1}t$ which conjugates A to $h^{-1}Bh$, so that $sa^j s^{-1} = a^i$ and hence $|i| = |j|$ is a necessary condition for G to be balanced, so we assume this for now on. By Proposition 6.3 the only elements we need to show are balanced are those in H having a power conjugate in H into $A \cup h^{-1}Bh$, so without loss of generality we take $h \in H$ with a power conjugate in H into A , but then we need only show that this power of a is unimodular and this reduces to looking at $I_G(a)$ which we will now show is equal to $\langle s, I_H(a) \rangle$.

As $s \in I_G(a)$ already, we again take $g = h_r s^{\epsilon_r} \dots s^{\epsilon_1} h_0 \in G \setminus H$ and in reduced form, along with an arbitrary power a^k of a . As before $ga^k g^{-1}$ will be reduced unless at least $\epsilon_1 = +1$ and $h_0 a^k h_0^{-1} \in A$, in which case it is equal to $a^{\pm k}$ as a is unimodular in H , or $\epsilon_1 = -1$ and $h_0 a^k h_0^{-1} \in h^{-1}Bh$. But the former case means that

$$s^{\epsilon_1} h_0 a^k h_0^{-1} s^{-\epsilon_1} = h^{-1} b^{\pm k} h$$

whereas if we had $h_0 a^k h_0^{-1} = h^{-1} b^l h$ in the latter case then $h_0 a^{ik} h_0^{-1} = h^{-1} b^{il} h = a^{\pm il}$, so also $|k| = |l|$ and therefore $s^{\epsilon_1} h_0 a^k h_0^{-1} s^{-\epsilon_1} = t^{-1} b^{\pm k} t = a^{\pm k}$. Thus again we see that we move between $a^{\pm k}$ and $h^{-1} b^{\pm k} h$ as we conjugate, but if the latter is a power of a then it can only be $a^{\pm k}$ as above. Hence if $g \in I_G(a)$ then $ga^k g^{-1}$ can only equal $a^{\pm k}$. \square

We can now apply the above results to graphs of groups, but in order to use these repeatedly we need to know how conjugacy works in these cases.

7 Conjugacy in graphs of balanced groups with infinite cyclic edge groups

We assume the usual definition and standard facts about a finite graph of groups $G(\Gamma)$ with fundamental group G where the underlying graph Γ has vertices $V(\Gamma)$ and (unoriented) edges $E(\Gamma)$. We write G_v for the vertex group at $v \in V(\Gamma)$, whereas on taking an edge $e \in E(\Gamma)$ and giving it an orientation so that it travels from the vertex v_1 to v_2 (where possibly $v_1 = v_2$), we write G_{e-} for the inclusion of the edge group in G_{v_1} and G_{e+} for the inclusion into

G_{v_2} . We also assume here that all vertex groups are torsion free and all edge groups are infinite cyclic. We define a vertex element $g_v \in G_v$ of $G(\Gamma)$ to be an element that actually lies in some particular vertex group, not just one that is conjugate into it. (Strictly speaking this is not well defined but it is once a maximal subtree of Γ is specified.) It seems we need to know when two vertex elements are conjugate in G ; in fact it turns out we only ever need to know when they have powers that are conjugate in G . The most obvious way in which this could hold is if they lie in the same vertex group and have powers that are conjugate in this vertex group. Our next definition incorporates what we regard as the second most obvious way.

Definition 7.1 *Given a graph of groups $G(\Gamma)$ with torsion free vertex groups and infinite cyclic edge groups, along with vertex elements $g \in G_v$ and $g' \in G_{v'}$, we define a **conjugacy path** p from g to g' to be an oriented non empty edge path $v = v_0, v_1, \dots, v_n = v' \in V(\Gamma)$ traversed by edges $e_1, \dots, e_n \in E(\Gamma)$ for $n \geq 1$ with e_i given the orientation from v_{i-1} to v_i such that the following conditions hold:*

- (1) *Some power of g is conjugate in G_{v_0} into the edge subgroup $G_{e_1^-}$ of G_{v_0} .*
 - (2) *For each $i = 1, \dots, n-1$, some conjugate in G_{v_i} of the edge subgroup $G_{e_i^+}$ intersects the edge subgroup $G_{e_{i+1}^-}$ non trivially.*
 - (3) *Some power of g' is conjugate in G_{v_n} into the edge subgroup $G_{e_n^-}$ of G_{v_n} .*
- We say that the conjugacy path p is **reduced** and/or **closed** if the underlying edge path is reduced and/or closed.*

Note: Every conjugacy path gives rise to a unique non empty edge path, but conversely suppose we have a non empty edge path from v to v' and elements g, g' in G_v and $G_{v'}$ respectively. Then we say this edge path **induces** a conjugacy path from g to g' if (1), (2) and (3) all hold.

If a conjugacy path exists from g to g' then it is clear that some power of g and some power of g' are conjugate in G : certainly a power of g is conjugate into $G_{e_1^-}$ which is either equal to or conjugate in G to $G_{e_1^+}$, depending on whether we form an amalgamated free product or HNN extension over the edge e_1 . But some element of $G_{e_1^+}$ is conjugate in G into $G_{e_2^-}$, and by taking higher powers if necessary we can assume that this element is conjugate in G to a power of g . We then continue in this way until we reach an element of $G_{e_n^-}$, and some power of this will be conjugate to a power of g' .

Our next step is to show that in order to establish the existence of conjugacy paths between two vertex elements, we need only use reduced paths.

This is certainly not the case if the edge groups are not infinite cyclic: for instance consider the amalgamated free product $G = A *_C X$ where A is free on a, b and X is free on x, y , with C also a rank 2 free group and the amalgamation defined by identifying a^2 with x and b^2 with xyx^{-1} . Then a^2 and b^2 are conjugate in G but not in A , though any conjugacy path establishing this will need to leave A and then return, thus will not be reduced. In fact there is some literature on conjugacy in graphs of groups, even specialising in the case where edge groups are infinite cyclic. However this is usually geared towards the conjugacy problem or conjugacy separability. We have not seen the following results elsewhere, perhaps because they are only concerned with conjugacy of unspecified powers, rather than the elements themselves. Furthermore it will be shown not only that we need just consider reduced paths but that it is enough to consider paths that never pass through the same edge twice. This is important for applications because it means that there are only ever finitely many such paths to check.

Proposition 7.2 *If there exists a conjugacy path from $g \in G_{v_0}$ to $g' \in G_{v_n}$ then either $v_0 = v_n$ and g, g' have powers which are conjugate in this vertex group, or we can take the underlying edge path to be reduced: indeed we can assume that this edge path only traverses any unoriented edge at most once.*

Proof. With the notation in Definition 7.1, suppose that e_r and e_s (for $r < s$) in the underlying edge path p are the same edge, running from v_{r-1} to v_r . Then some conjugate in G_{v_r} of the edge subgroup $G_{e_r^+} = G_{e_s^+}$ intersects both edge subgroups $G_{e_{r+1}^-}$ and $G_{e_{s+1}^-}$ non trivially. In particular some conjugate in G_{v_r} of $G_{e_r^+}$ intersects $G_{e_{s+1}^-}$. Thus we can remove from p the edges e_{r+1}, \dots, e_s which run from v_r back to itself and we still have a conjugacy path from g to g' . We now continue until we have removed all such repeats.

If though e_s is the reverse of e_r , running backwards from v_r to v_{r-1} then the argument is similar but now we have $G_{e_r^-}$ equal to $G_{e_s^+}$ in $G_{v_{r-1}}$ so that there are conjugates in $G_{v_{r-1}}$ of $G_{e_{r-1}^+}$ and of $G_{e_{s+1}^-}$ which both intersect $G_{e_r^-} = G_{e_s^+}$. This time we can cut out the edges e_r, \dots, e_s and still have a conjugacy path from g to g' , unless we have cut out the whole path. In this case the edge e_r is actually e_1 from v_0 to v_1 and the edge $e_s = e_n$ is the reverse of e_1 , with a power of g (respectively g') conjugate in G_{v_0} (respectively G_{v_n} which is equal to G_{v_0}) into $G_{e_1^-}$ (respectively $G_{e_n^+}$ which is equal to $G_{e_1^-}$). Thus there are powers of g and g' which are already conjugate in G_{v_0} . \square

We can now consider when there exist powers of two vertex elements which are conjugate in the fundamental group of a graph of torsion free groups with infinite cyclic edge groups. We start when the graph is a tree, though we first note the conjugacy theorem for amalgamated free products in [13]. Here we only require a partial version which can easily be proven by using reduced forms as in the previous section.

Proposition 7.3 *Let $G = A *_C B$ be an amalgamated free product and let $g \in A$.*

If $g' \in A \cup B$ and g, g' are conjugate in G but g is not conjugate in A into C then $g' \in A$ with g, g' conjugate in A .

Theorem 7.4 *Let $G(\Gamma)$ be a finite graph of torsion free groups with infinite cyclic edge groups and where Γ is a tree. Suppose we have two vertex elements $g \in G_v$ and $g' \in G_{v'}$. Then some power of g is conjugate in G to some power of g' if and only if either*

- (i) The vertices v, v' are equal and some power of g is conjugate in G_v to some power of g' .*
- (ii) The vertices v, v' are distinct and the unique reduced path in Γ from v to v' induces a conjugacy path from g to g' .*

Proof. We have seen that these conditions are sufficient for conjugacy so we prove necessity by induction on the number of edges in Γ , with Proposition 7.3 being our base case: if $G = A *_C B$ has one edge and g, g' are both in A with a power of g conjugate in G to a power of g' then either these powers are conjugate in A , or they are both conjugate in A into C and so further powers of g and g' are conjugate in A to each other anyway. Alternatively if $g \in A$ but $g' \in B$ and g^i is conjugate in G to g'^j then g^i is conjugate in A into C and g'^j conjugate in B into C , giving us our conjugacy path from g to g' .

In the general case we again suppose that some power of g is conjugate in G to some power of g' . First suppose that $v = v'$. We remove a leaf vertex $w \neq v$ in Γ and its adjoining edge e_w from Γ to form the tree Γ_w . Now we can form the fundamental group A of the graph of groups given by the tree Γ_w , which means that $G = A *_C G_w$, where C is the infinite cyclic edge group G_{e_w} . Now as $v \in \Gamma_w$ we have that $g, g' \in A$. But, as in the base case, if both a power of g and a power of g' are conjugate in A into C then some power of g is conjugate in A to some power of g' . If not then g say has no power conjugate in A into C , in which case we have by Proposition 7.3 that

any element of A which is conjugate in $G = A *_C B$ into $\langle g \rangle$ must already be conjugate in A . Either way we are now conjugate back in A so the induction holds to obtain conjugates of a power of g and a power of g' back in G_v .

To prove (ii), first suppose that no power of g is conjugate in G_v into the edge inclusion G_{e_1-} . We similarly remove the edge e_1 from Γ to form trees Γ_0, Γ_1 and the amalgamated free product $A *_C B$, where A is the fundamental group of the graph of groups obtained from Γ_0 , B from Γ_1 and $C = G_{e_1-}$. Hence all powers of g are in A and all powers of g' are in B , but we can only have the element g^i of A conjugate to an element of B if g^i is conjugate in A into $C = \langle c \rangle$. As the graph Γ_0 is a tree, we can use induction to determine if a power of $g \in G_v$ is conjugate in A to a power of $c \in G_v$, whereupon we see this occurs if and only if these powers were already conjugate in G_v .

We then argue in the same way if no power of g' is conjugate in $G_{v'}$ into the subgroup G_{e_n+} of $G_{v'}$. Otherwise the only way that (ii) can fail is if somewhere along this edge path joining v to v' , say at the vertex v_i , we have that no conjugate of G_{e_i+} in G_{v_i} meets $G_{e_{i+1}-}$, and we suppose that this is the first time it occurs. We remove the edge e_{i+1} giving us the decomposition $G = A *_C B$ where C embeds as $G_{e_{i+1}-}$ in A and as $G_{e_{i+1}+}$ in B . Now some power of g is this time conjugate in G_{v_0} (for $v_0 = v$) into G_{e_1-} , and hence by following the path from v_0 up to v_i , we have that a power of g is conjugate in A into the edge inclusion G_{e_i+} . Now suppose that some other power of g is conjugate in G into $\langle g' \rangle \leq B$. By Proposition 7.3 this can only happen if this new power of g is conjugate in A into $C = G_{e_{i+1}-}$. But this would force a further power of g to be conjugate in A both into G_{e_i+} and into $G_{e_{i+1}-}$, hence G_{e_i+} can be conjugated in A to meet $G_{e_{i+1}-}$, and hence by induction also in G_{v_i} which is a contradiction. \square

We now need to provide a similar result for general graphs of groups. Thus we require the equivalent version of Proposition 7.3 for HNN extensions, usually known as Collins' criterion. Again we just require the simplified version below which can be verified with reduced forms.

Proposition 7.5 *Suppose that G is an HNN extension of H with stable letter t and associated subgroups A, B of H so that $tAt^{-1} = B$ and let $g \in H$ be conjugate in G to $g' \in H$. Then*
If g is not conjugate in H into $A \cup B$ then nor is g' and g is conjugate in H

to g' .

Theorem 7.6 *Let $G(\Gamma)$ be a finite graph of torsion free groups with infinite cyclic edge groups. Suppose we have two vertex elements $g \in G_v$ and $g' \in G_{v'}$. Then some power of g is conjugate in G to some power of g' if and only if either:*

- (i) *The vertices v, v' are equal and some power of g is conjugate in G_v to some power of g' .*
- (ii) *There exists some conjugacy path from g to g' (where closed paths with $v = v'$ are allowed).*

Proof. Again these conditions clearly imply conjugacy so we suppose that a power of g and a power of g' are conjugate in G . We have our result when Γ is a tree, so we now take a maximal tree T in Γ and argue by induction on the number of remaining edges, with the base case being Theorem 7.4.

Suppose on removing from Γ an edge e not in T we are left with the connected graph Δ , giving rise to the graph of groups $H(\Delta)$ so that G is formed from H by an HNN extension with associated subgroups $A = G_{e^-} \leq G_w \leq H$ and $G_{e^+} \leq G_{w'} \leq H$, where e runs from the vertex w to w' (where we could have $w = w'$). Again we ask: is this power of g conjugate in H into $A \cup B$? If not then our powers of g and g' must be conjugate in H by Proposition 7.5, and so we can inductively use the criterion of conjugacy in $H(\Delta)$ instead. Otherwise this power is conjugate in H to some element a say of A without loss of generality, and we must also have in this case that our power of g' is conjugate in H into either A or into B by Proposition 7.5 with g and g' swapped. If it is $A = G_{e^-}$ then as before some other power of g will be conjugate in H to some power of g' so that we are back with the inductive statement for $H(\Delta)$.

If however this power of g' is conjugate in H to $b \in B$ say then we can at least use the induction to say that there is a conjugacy path in $H(\Delta)$, and hence in $G(\Gamma)$, from g to a which joins the vertices v and w , as well as a conjugacy path from g' to b joining v' and w' (or we have conjugacy of powers within the relevant vertex groups). But as the edge e induces a conjugacy path in $G(\Gamma)$ from a to b , we can put these together to get one from g to g' . \square

8 Graphs of balanced groups

We can now put together the results of the previous two sections.

Definition 8.1 *Suppose that $G(\Gamma)$ is a finite graph of balanced, torsion free groups with infinite cyclic edge groups. We say a reduced, closed, conjugacy path e_1, \dots, e_n from $g \in G_v$ to g' in the same vertex group G_v , where g generates the edge group inclusion $G_{e_1^-} \leq G_v$ and g' generates $G_{e_n^+} \leq G_v$ is **complete** if some powers of g and g' are themselves conjugate in G_v . If so then, as the existence of the conjugacy path implies that some power of g is conjugate in G to some power of g' , we obtain two powers g^i, g^j of g which are conjugate in G . We say that our complete conjugacy path is **level** if $|i| = |j|$. This is well defined as all vertex groups are balanced and it does not depend on which power g^i we initially take provided it can be conjugated all the way round the conjugacy path.*

Proposition 8.2 *Given a finite graph $G(\Gamma)$ of balanced, torsion free groups with infinite cyclic edge groups and a complete conjugacy path from $g \in G_v$ to $g' \in G_v$, there exists a complete conjugacy path which passes through every non oriented edge of Γ at most once, though possibly starting (and thus ending) at a different vertex. If our original conjugacy path is not level then we can arrange that this new path is not level either.*

Proof. Apply the method in the proof of Proposition 7.2, so that if e_r and e_s (for $r < s$) are the same oriented edge then we create two conjugacy paths $e_1, \dots, e_r, e_{s+1}, \dots, e_n$ and e_{r+1}, \dots, e_s which are both closed, reduced and complete. If both paths are level then on putting these together, so was the original path.

The same argument works if e_s is the reverse of e_r for r minimal with this property and this will create the two new paths $e_1, \dots, e_{r-1}, e_{s+1}, \dots, e_n$ and e_{r+1}, \dots, e_{s-1} , except that the first edge path might be empty if $r = 1$ and $s = n$ (though the second will never be as complete conjugacy paths are reduced). But if so then the fact that the original path is not level implies that the second path is not either.

□

Theorem 8.3 *Given a finite graph $G(\Gamma)$ of balanced, torsion free groups with infinite cyclic edge groups, we have that the fundamental group G is not*

balanced if and only if there exists a complete non level conjugacy path in $G(\Gamma)$. By Proposition 8.2 this path can be taken to pass through any unoriented edge at most once and so, as the underlying edge path will determine a complete conjugacy path including the initial and final elements (or their inverses), there are only finitely many paths to check.

Proof. Clearly a complete non level conjugacy path gives rise to an unbalanced element, thus let us assume all complete conjugacy paths are level and again take a maximal tree T of Γ . On forming the graph of groups $H(T)$ we have that this fundamental group H is balanced by Corollary 6.2. Now suppose that $\Gamma \setminus T$ consists of the edges e_1, \dots, e_n . On inserting the edge e_1 to form the graph of groups $G_1(\Gamma_1)$, we have that G_1 is an HNN extension $\langle H, t_1 \rangle$. Let us suppose that e_1 runs from the vertex v_1 to w_1 , so that the associated subgroups of this HNN extension are $\langle a \rangle = G_{e_1^-}$ in G_{v_1} and $\langle b \rangle = G_{e_1^+}$ in G_{w_1} with $t_1 a t_1^{-1} = b$. By Proposition 6.3 we have that G_1 is still balanced if no conjugate in H of $\langle b \rangle$ intersects $\langle a \rangle$ non trivially, so now suppose that there is $h \in H$ so that $h a^i h^{-1} = b^j$. By Theorem 7.4 applied to $H(T)$, this can only happen if our relation is obtained from a reduced conjugacy path in $H(T)$ from a to b that joins v_1 to w_1 , or if $v_1 = w_1$ and there is $\gamma \in G_v$ with $\gamma a^k \gamma^{-1} = b^l$. But then on adding the edge e_1 with stable letter t_1 , we have a conjugacy path from a back to itself where $h^{-1} t_1$ (or $\gamma^{-1} t_1$) conjugates a^j to a^i (or a^l to a^k) and which is reduced and therefore complete. Thus if $|j| \neq |i|$ (or $|l| \neq |k|$) then this conjugacy path is not level and G_1 , which will be a subgroup of G , is not balanced. If however $|i| = |j|$ (or $|k| = |l|$) then the path is level and G_1 is balanced by Theorem 6.4.

We can then form further HNN extensions G_2, G_3, \dots by adding an edge each time until we reach the fundamental group G and the above argument applies at each stage, but in place of Theorem 7.4 we use Theorem 7.6 as well as Proposition 7.2 to ensure any conjugacy path used is reduced. \square

In particular, for any graph of groups as in this theorem which has a complete non level conjugacy path, the resulting fundamental group cannot be subgroup separable and cannot be linear over \mathbb{Z} . Moreover this fundamental group cannot lie in any of the classes of balanced groups given in Section 2, even if all the vertex groups lie in such a class. In [17] Theorem 5.1 it was shown using geometric means that a finite graph of groups with all vertex groups free and all edge groups infinite cyclic is subgroup separable if and only if it is balanced. It is also mentioned that, as a consequence of this

proof, the unbalanced case can be seen from the graph of groups and the corresponding description fits with Theorem 8.3.

9 Presence of Baumslag Solitar groups

Although we saw in Section 3 that an unbalanced torsion free group need not contain a Baumslag Solitar group, we would now like conditions on the vertex groups of a graph of torsion free groups which ensure that if the fundamental group is unbalanced then it contains a non Euclidean Baumslag Solitar subgroup. Although we do not show this for all torsion free groups, we would at least like to ensure that it holds when the vertex groups are all word hyperbolic or all free abelian, or even any combination of these. The following condition allows a considerable generalisation of this case.

Definition 9.1 *We say a torsion free group G has the **cohomological condition** if whenever we have a, b non trivial elements of G such that $\langle a \rangle \cap \langle b \rangle$ is non trivial, the subgroup $\langle a, b \rangle$ of G has cohomological dimension at most 2.*

Proposition 9.2 *The following groups have the cohomological condition.*

- (0) *Subgroups of groups with the cohomological condition*
- (i) *Groups of cohomological dimension at most 2*
- (ii) *Torsion free word hyperbolic groups*
- (iii) *More generally, torsion free groups which are CT*
- (iv) *Torsion free abelian groups, or more generally torsion free nilpotent groups*
- (v) *More generally still, groups that are residually (torsion free nilpotent)*
- (vi) *Torsion free groups which are relatively hyperbolic with respect to subgroups that satisfy the cohomological condition.*

Proof. (0) is obvious, as is (i) by Shapiro's Lemma that a subgroup cannot increase cohomological dimension. For (iii) if we have $a, b \in G$ with powers $a^r = b^s$ then a and b commute with this element, thus with each other and so $\langle a, b \rangle$ is torsion free abelian, thus is \mathbb{Z} . For (iv) we recall that a finitely generated torsion free nilpotent group is either trivial, \mathbb{Z} or it surjects to \mathbb{Z}^2 (for instance it is residually finite- p for every prime p by a result of Gruenberg, thus it must surject $(C_p)^2$ or be cyclic) and being torsion free nilpotent is preserved on passing to subgroups. But if $a, b \in G$ with powers a^r equal to

b^s then $\langle a, b \rangle$ cannot surject to \mathbb{Z}^2 . For (v), being residually (torsion free nilpotent) is also preserved by subgroups, so if G is residually (torsion free nilpotent), thus torsion free, and we have elements a, b in G with $a^r = b^s$ but $\langle a, b \rangle$ is not abelian (otherwise it is \mathbb{Z}) then $\langle a, b \rangle$ surjects to a non abelian torsion free nilpotent group. This must surject to \mathbb{Z}^2 thus so does $\langle a, b \rangle$.

Finally if G is relatively hyperbolic with respect to subgroups having this condition and we have elements a, b of G with $a^r = b^s$ then it follows immediately from Lemma 5.1 on taking $g = a^r = b^s$ that either $\langle a, b \rangle$ is infinite cyclic or it can be conjugated into a peripheral subgroup. \square

Groups of cohomological dimension 2 need not be balanced, thus groups which are relatively hyperbolic with respect to these need not be either, but we saw earlier that groups in all the other categories will be, apart from torsion free CT groups where the exceptions were identified in Proposition 3.7.

We will want to show that if all vertex groups have this property and are balanced then an unbalanced graph of groups with infinite cyclic edge groups contains a Baumslag Solitar subgroup that is non Euclidean. We start by mentioning a couple of well known lemmas: the first following from Mayer-Vietoris considerations and the second from the usual use of reduced forms.

Lemma 9.3 *If $G(\Gamma)$ is a finite graph of groups with all vertex groups having cohomological dimension at most 2 and all edge groups are infinite cyclic then G has cohomological dimension at most 2.*

Lemma 9.4 (i) *If $G = A *_C B$ is an amalgamated free product and we have subgroups $A' \leq A$ and $B' \leq B$ which both contain C then the subgroup $\langle A', B' \rangle$ of G can be expressed naturally as the amalgamated free product $A' *_C B'$.*

(ii) *Suppose that $G = \langle H, t \rangle$ is an HNN extension with base H , stable letter t and associated subgroups A, B such that $tAt^{-1} = B$.*

If we have a subgroup R of H which contains both A and B then the subgroup $\langle R, t \rangle$ of G is naturally the HNN extension with base R , stable letter t and associated subgroups A, B such that $tAt^{-1} = B$.

*If we have subgroups J, L of H with J containing A and L containing B then the subgroup $\langle tJt^{-1}, L \rangle$ of G can be naturally expressed as the amalgamated free product $tJt^{-1} *_C L$.*

We now transfer the cohomological property from the vertex groups to the fundamental group of the graph of groups.

Theorem 9.5 *Let $G(\Gamma)$ be a finite graph of groups with all vertex groups satisfying the cohomological condition and all edge groups infinite cyclic. Suppose we have a conjugacy path between vertex elements a, b of G , thus providing us with an element $g \in G$ and $i, j \neq 0$ such that $ga^i g^{-1} = b^j$. Then $\langle gag^{-1}, b \rangle$ has cohomological dimension at most 2.*

Proof. In the proof that follows, cohomological dimension 2 will actually stand for cohomological dimension at most 2. We first assume that Γ is a tree T , so that we are in the same set up as Theorem 7.4 Case (ii). Then we have our edge path e_1, \dots, e_n running from $A = G_{v_0}$ containing a to $B = G_{v_n}$ containing b , with the edge e_k running from the vertex v_{k-1} with vertex group $G_{v_{k-1}}$ to v_k with vertex group G_{v_k} . We set $\langle f_k \rangle$ equal to the edge group G_{e_k} and use this notation for its image in both of the neighbouring vertex groups.

Writing out in order the conjugation equalities that hold in each vertex group, we obtain

$$g_0 a^{i_1} g_0^{-1} = f_1^{j_1}, g_1 f_1^{i_2} g_1^{-1} = f_2^{j_2}, \dots, g_{n-1} f_{n-1}^{i_n} g_{n-1}^{-1} = f_n^{j_n}, g_n f_n^{i_{n+1}} g_n^{-1} = b^{j_{n+1}}$$

where g_k is the conjugating element in the vertex group G_{v_k} so that $g = g_n g_{n-1} \dots g_1 g_0$, $f_k \in G_{v_{k-1}} \cap G_{v_k}$ and i_1, \dots, i_{n+1} and j_1, \dots, j_{n+1} are integers such that $i_1 i_2 \dots i_{n+1} = i$ and $j_1 j_2 \dots j_{n+1} = j$. Let us set A_0 to be the subgroup $\langle g_0 a g_0^{-1}, f_1 \rangle$ of G_{v_0} and B_0 to be the subgroup $\langle f_1, g_1^{-1} f_2 g_1 \rangle$ of G_{v_1} . By the cohomological condition on the vertex subgroups and the conjugation equalities above, both A_0 and B_0 have cohomological dimension 2, thus $\langle g_0 a g_0^{-1}, f_1 \rangle *_{\langle f_1 \rangle} \langle f_1, g_1^{-1} f_2 g_1 \rangle$ does too. But by Lemma 9.4 (i) this is equal to the subgroup $H_1 = \langle g_0 a g_0^{-1}, f_1, g_1^{-1} f_2 g_1 \rangle$ of $S_1 = G_{v_0} *_{\langle f_1 \rangle} G_{v_1}$, so the subgroup $\langle g_1 g_0 a (g_1 g_0)^{-1}, f_2 \rangle$ of $g_1 H_1 g_1^{-1}$ also has cohomological dimension 2. We can now amalgamate this with the cohomological dimension 2 subgroup $\langle f_2, g_2^{-1} f_3 g_2 \rangle$ of G_{v_2} over the subgroup $\langle f_2 \rangle$ which results in a cohomological dimension 2 subgroup of $S_2 = g_1 S_1 g_1^{-1} *_{\langle f_2 \rangle} G_{v_2}$ and so on, until we conclude that $\langle g_{n-1} \dots g_1 g_0 a (g_{n-1} \dots g_1 g_0)^{-1}, g_n^{-1} b g_n \rangle$ has cohomological dimension 2, thus so does the conjugate subgroup $\langle gag^{-1}, b \rangle$.

Now we move to the case where the graph Γ is not a tree, so we take a maximal tree T in Γ with fundamental group H . Again we have a conjugacy path from a to b and we are in the same situation as for the tree except that this time we might walk over edges e_k that are not in T , in which

case our conjugating equality $g_{k-1}f_{k-1}^{i_k}g_{k-1}^{-1} = f_k^{j_k}$ is still the same, but the following conjugating equality which was previously $g_k f_k^{i_{k+1}} g_k^{-1} = f_{k+1}^{j_{k+1}}$ is now $g_k t f_k^{i_{k+1}} t^{-1} g_k^{-1} = f_{k+1}^{j_{k+1}}$ for t the relevant stable letter (or the inverse thereof) associated to the edge e_k . However we know that we only walk over such an edge once by Proposition 7.2.

Let us start by supposing that e_k is the first edge walked over in $\Gamma \setminus T$. The situation now is that we would already know the subgroup

$$\langle g_{k-1} \dots g_1 g_0 a(g_{k-1} \dots g_1 g_0)^{-1}, f_k \rangle$$

of H has cohomological dimension 2 by the same argument as for the tree. Now we can use our edge e_k to form the HNN extension $G_1 = \langle H, t \rangle$, although this time the edge subgroups will be written as $\langle f_k \rangle \leq G_{v_{k-1}}$ and $\langle t f_k t^{-1} \rangle \leq G_{v_k}$, so that the conjugate subgroup $\langle t g_{k-1} \dots g_1 g_0 a(g_{k-1} \dots g_1 g_0)^{-1} t^{-1}, t f_k t^{-1} \rangle$ has cohomological dimension 2 as well.

We also consider the subgroup

$$H_k = \langle t g_{k-1} \dots g_1 g_0 a(g_{k-1} \dots g_1 g_0)^{-1} t^{-1}, t f_k t^{-1}, g_k^{-1} f_{k+1} g_k \rangle,$$

of G_1 which is equal to

$$\langle t g_{k-1} \dots g_1 g_0 a(g_{k-1} \dots g_1 g_0)^{-1} t^{-1}, t f_k t^{-1} \rangle *_{\langle t f_k t^{-1} \rangle} \langle t f_k t^{-1}, g_k^{-1} f_{k+1} g_k \rangle$$

by Lemma 9.4 (ii) where

$$A = \langle f_k \rangle, B = \langle t f_k t^{-1} \rangle, J = \langle g_{k-1} \dots g_1 g_0 a(g_{k-1} \dots g_1 g_0)^{-1}, f_k \rangle,$$

and $L = \langle t f_k t^{-1}, g_k^{-1} f_{k+1} g_k \rangle$, with L also having cohomological dimension 2 because $g_k L g_k^{-1}$ is a subgroup of G_{v_k} with powers of its two generators equal by the second conjugacy inequality above. Thus on applying Lemma 9.3 again, we have that H_k has cohomological dimension 2 and therefore so does

$$\langle g_k t g_{k-1} \dots g_1 g_0 a(g_k t g_{k-1} \dots g_1 g_0)^{-1}, f_{k+1} \rangle$$

as it is conjugate in G_1 into a subgroup of H_k .

If this is the only edge in the conjugacy path that lies outside T then the remainder of the proof is as above, because we now walk over the remaining edges e_{k+1}, \dots, e_n which all lie in T . If however there are further edges outside T then we can build up the fundamental group G from H by a sequence of HNN extensions which we perform in the order we walk over

them (and then arbitrarily for any edges left over). Our proof now works as before except that a stable letter will appear within the product $g_n \dots g_1 g_0$ whenever we walk over an edge in $\Gamma \setminus T$.

□

We can now obtain the existence of Baumslag-Solitar subgroups by reducing to known facts about groups of cohomological dimension 2.

Corollary 9.6 *If $G(\Gamma)$ is a finite graph of groups with infinite cyclic edge groups and where every vertex group is torsion free, balanced, and satisfies the cohomological condition then the fundamental group G is not balanced exactly when $G(\Gamma)$ contains a complete non level conjugacy path, which is exactly when G contains a non Euclidean Baumslag Solitar subgroup.*

Proof. If G is balanced then so are its subgroups and by Theorem 8.3, G is not balanced exactly when we have such a path. If so then consider the construction in this proof where we add edges one by one to a maximal tree and let us stop on the first occasion where the resulting fundamental group is not balanced. This group will end up being a subgroup of G so we now replace the final graph of groups $G(\Gamma)$ with this one. Thus on taking $H(\Delta)$ to be the graph of groups immediately before this edge was added, we now have $G = \langle H, t \rangle$ and vertex elements $a, b \in H$ such that G is the HNN extension with base H and $tat^{-1} = b$. But as this edge lies in a complete non level conjugacy path, following the rest of the path implies that we obtain $h \in H$ with $ha^i h^{-1} = b^j$ for i, j with $|i| \neq |j|$. Thus application of Theorem 9.5 with $G(\Gamma)$ now equal to $H(\Delta)$ and G now equal to H tells us that $\langle hah^{-1}, b \rangle$ has cohomological dimension at most 2. Now on replacing the stable letter t with the alternative stable letter $s = h^{-1}t$, we have that G is also the HNN extension with base H and associated subgroups $\langle a \rangle$ and $\langle h^{-1}bh \rangle$ with s conjugating a to $c = h^{-1}bh$. Thus by Lemma 9.4 the subgroup $S = \langle a, c, s \rangle$ is also an HNN extension with base $\langle a, c \rangle$ and the same associated subgroups. As the base has cohomological dimension 2, so does this HNN extension S in which $sa^j s^{-1} = a^i$ holds for $|i| \neq |j|$. This does not imply that $S = \langle s, a \rangle$ is isomorphic to $BS(j, i)$ but as $I_S(a) = S$ which is of cohomological dimension 2, the main result of [10] tells us that S is a generalised Baumslag-Solitar group and we see that $r = j/i \neq \pm 1$ is the image of the elliptic element a under the modular homomorphism of S . This implies by [11] Proposition 7.5 that S contains a subgroup isomorphic to $BS(m, n)$, where m/n is the expression of r in lowest terms, and so $BS(m, n)$ is non Euclidean.

If instead the fundamental group G contains a non Euclidean Baumslag-Solitar subgroup then this subgroup and hence G itself is not balanced, so we are covered by Theorem 8.3.

□

Thus for instance we have that a finite graph of groups with free vertex groups and infinite cyclic edge groups is balanced if and only if it does not contain a non Euclidean Baumslag - Solitar group, which was not explicitly stated in [17].

10 Hyperbolic graphs of groups

In a torsion free hyperbolic group H , an element h is called maximal if whenever $h = a^i$ for $a \in H$ we have $i = \pm 1$. However for these groups this condition is equivalent to h generating its own centraliser (or even its own intersector). Here centralisers are always infinite cyclic and moreover (on removal of the identity) they partition H into infinite cyclic subgroups which are all maximal. In this section we will nearly always be dealing with torsion free word hyperbolic groups, but if not then saying an element is maximal will actually mean here that it generates its own centraliser.

We have the Bestvina - Feighn theorem for hyperbolicity of amalgamated free products with infinite cyclic edge groups:

Lemma 10.1 ([3] Section 7, second Corollary) *Suppose that $G = A *_C B$ is an amalgamated free product where $C = \langle c \rangle$ is infinite cyclic and A, B are torsion free word hyperbolic groups. Then G is word hyperbolic if and only if c is maximal in one of A or B , which occurs if and only if G does not contain \mathbb{Z}^2 .*

The following result can be deduced directly from this along with use of reduced forms, but by considering centralisers rather than the maximal elements themselves.

Lemma 10.2 *Suppose that $G = A *_C B$ is an amalgamated free product where $C = \langle c \rangle$ is infinite cyclic and A, B are torsion free word hyperbolic groups. Let us take an element $a \in A$. Then*

- (i) *If a is not maximal in A it is clearly not maximal in G .*
- (ii) *If a is maximal in A but is not conjugate in A into C then a is still maximal in G .*

Thus suppose from now on that a is maximal in A and conjugate in A into C , thus to exactly one of c and c^{-1} so say c without loss of generality. In particular c is maximal in A and thus G is word hyperbolic by Lemma 10.1.

(iii) Suppose that c is also maximal in B then c is still maximal in G .

(iv) Suppose that c is not maximal in B then any element conjugate in A to $c^{\pm 1}$ is clearly maximal in A but not maximal in G .

We now introduce a similar notion to the conjugacy paths already considered, in order to keep track of vertex elements which are maximal in their vertex group but not in the fundamental group. Given an unoriented edge e in the graph of groups $G(\Gamma)$ with infinite cyclic edge groups and torsion free hyperbolic vertex groups, we consider the inclusions of the edge group G_e in its neighbouring vertex groups G_v and G_w (where maybe $v = w$). If G_e is included in G_v as a non maximal subgroup then we put an arrow on e at the end of e next to v and we make this arrow point towards v . If G_e is included maximally in G_v then no arrow is added, and we then do the same with G_w and then over all edges in the graph.

Definition 10.3 Given vertex elements $g_v \in G_v$ and $g_w \in G_w$ (where maybe $v = w$), a **semi non maximal path** from g_v to g_w is a conjugacy path in $G(\Gamma)$ from g_v to g_w such that no edges in this path are labelled with arrows as above, apart from a single arrow on the initial edge which points towards the vertex v .

A **full non maximal path** from g_v to g_w is a reduced conjugacy path in $G(\Gamma)$ from g_v to g_w such that all edges in this path are unlabelled, apart from an arrow on the initial edge which points towards the vertex v and an arrow on the final edge which points towards w , and such that the final edge is not the reverse of the initial edge.

Proposition 10.4 Suppose we have a finite graph $G(\Gamma)$ of torsion free word hyperbolic groups with infinite cyclic edge groups. If there exists a semi non maximal path from $g \in G_{v_0}$ to $g' \in G_{v_n}$ then we can replace it with one that also runs from $g \in G_{v_0}$ to $g' \in G_{v_n}$ but which traverses any unoriented edge at most once. The same statement is true for any full maximal path.

Proof. We again run through the proof of Proposition 7.2, noting that the initial edge of any semi maximal path is the only labelled edge, thus this will remain and only unlabelled edges will be removed. Similarly the only labelled edges in a full maximal path are the initial and final ones, but these

cannot be the same oriented edge as the arrows point in different directions (unless the path is just this single edge) and the definition rules out them being the reverse of each other. Thus again only unlabelled edges can be removed.

□

We can use semi maximal paths to determine maximal elements in a graph of groups: suppose there is one from $g_v \in G_v$ to $g_w \in G_w$ where g_v and g_w are both maximal in their respective vertex groups. If this path is considered just as a conjugacy path, it would only indicate that some powers of g_v and of g_w are conjugate in G . But as every edge apart from the initial one is unmarked, all generators of these edge groups include into all subsequent vertex groups as maximal elements, and therefore we see that actually we have g_v^i and g_w are conjugate in G for $|i| > 1$ so that g_w is not maximal in G , whether or not g_v is.

We now consider when such a graph of groups has a fundamental group which is word hyperbolic, starting with a tree.

Theorem 10.5 *Suppose that $G(\Gamma)$ is a graph of groups where Γ is a tree, with infinite cyclic edge groups and all vertex groups torsion free word hyperbolic. Then G is word hyperbolic unless there exists a full non maximal path in $G(\Gamma)$, in which case G contains \mathbb{Z}^2 , and if so then there exists such a path passing through any non oriented edge at most once by Proposition 10.4 thus there are only finitely many paths to check.*

Proof. As before, the proof is by induction on the number of edges. However we also need to keep track of maximal elements so our inductive statement is as follows:

- (i) If there exists a full non maximal path in $G(\Gamma)$ then G contains \mathbb{Z}^2 and so is not word hyperbolic.
- (ii) If $G(\Gamma)$ contains no full non maximal paths then G is word hyperbolic.
- (iii) If G is word hyperbolic then a vertex element g_v which is maximal in its vertex group is non maximal in G if and only if there exists a semi non maximal path that ends at g_v .

Lemmas 10.1 and 10.2 give us the base case. Now given the graph of groups $A(T)$, let us add an edge e to the tree T to form the tree Γ with e having the vertex $v \in T$ and $w \notin T$, so that the fundamental group G of $G(\Gamma)$ is equal to $A *_C B$ where B is the vertex group G_w .

By the inductive hypothesis applied to $A(T)$, if there were a full non maximal path in $A(T)$ and thus in $G(\Gamma)$ then A and G would contain \mathbb{Z}^2 , so (i), (ii), (iii) hold for $G(\Gamma)$ in this case. Hence now we assume that no such path exists in $A(T)$ and consequently by induction that A is torsion free word hyperbolic.

First suppose that there is no arrow on e pointing towards the vertex w , so that $C = \langle c \rangle$ is included maximally in B . Then no full maximal path can lie in $G(\Gamma)$ that does not already lie in $A(T)$ (as these are reduced paths by definition), so (i) holds in this case. Moreover as we already have maximality on one side, we obtain (ii) because G is word hyperbolic by [3]. As for (iii), on application of Lemma 10.2 (with A and B as they are and then swapped), we see that the only maximal elements of A and B that might no longer be maximal in G are those maximal elements b of B which are conjugate in B into C , and then only if c is not a maximal element of A . But this latter case can only happen if either c is not a maximal element in the vertex group G_v , in which case e has an arrow pointing towards v and we have our semi non maximal path from $c \in G_v$ to $c \in G_w = B$, or by the inductive hypothesis we have a semi non maximal path in $A(T)$ that starts somewhere else and ends at $c \in G_v$, in which case we add the edge e to the end of this path to get one that ends at $b \in B = G_w$. Conversely the only way that new semi non maximal paths can be created in $G(\Gamma)$ is by using e itself, either as the only edge if it has an arrow pointing towards v , or by adding it on if no arrow is present. If this is not the final edge then we immediately need to backtrack, thus Proposition 10.4 tells us that we can regard it as lying in $A(T)$ anyway.

For our second case, we suppose that there is an arrow on e pointing towards w , so that in $B = G_w$ we have $c = d^i$ say, for d maximal in B and $|i| > 1$. If there is another arrow on e pointing towards v then we immediately have a full non maximal path in $G(\Gamma)$, with G non hyperbolic and containing \mathbb{Z}^2 by [3] because neither edge inclusion is maximal. Otherwise c is a maximal element of G_v which might not or might be a maximal element of A . By part (iii) of the inductive hypothesis applied to A , which is assumed to be word hyperbolic, this is determined by the existence or not of a semi non maximal path in $A(T)$ which ends at $c \in G_v$ (assumed reduced by Proposition 10.4). But if one exists then by adding the edge e to the end, we have a full non maximal path in $G(\Gamma)$ with $G = A *_C B$ being non maximal on both sides, thus again we have \mathbb{Z}^2 in G and (i) is confirmed in this case too. If however there are no semi non maximal paths in $A(T)$ that end at $c \in G_v$ then we can assume that c is maximal in A and thus $G = A *_C B$ is word hyperbolic,

giving (ii) here.

Finally we need to establish (iii) in the second case, whereupon we are now dealing with c being maximal in A as well as in G_v . But c is clearly not maximal in G and so again by Lemma 10.2 the “new non maximal vertex elements” are exactly those vertex elements that are maximal in A and are conjugate in A to $c^{\pm 1} \in G_v$. Now a conjugacy path in $A(T)$ from such an element $g_u \in G_u$ for $u \in T$ to $c \in G_v$ as in Section 7 is clearly a necessary condition for this to occur. Let us take such a path (travelling over any edge at most once) and first look at where arrows might appear on these edges. If we start at $g_u \in G_u$ and walk towards $c \in G_w$, suppose that along the way we encounter an arrow pointing in the opposite direction of travel. As we end up at w with an arrow pointing along with us, on finding the final arrow encountered that points in the reverse direction we obtain a full non maximal path in $G(\Gamma)$, at least on application of Proposition 10.4. But we have already established part (i) of our inductive statement for $G(\Gamma)$ so this is a contradiction.

If however we encounter an arrow along the way that points in our direction then the reverse of this path so far is already a semi non maximal path lying in $A(T)$ and ending at g_u , so by the inductive hypothesis g_u was not maximal in A . Thus our conjugacy path has no arrows appearing and thus by adding on the edge e at the end and reversing, we are left with a semi non maximal path running from $c \in B$ to $g_u \in G_u$. As for the converse, the new semi non maximal paths will all be created by starting at $b \in G_w$ with the edge e and then following a conjugacy path in $A(T)$ which has no edges labelled, which certainly means that the end element is not maximal in G . \square

We now move to HNN extensions, where [4] gives the exact conditions for word hyperbolicity over virtually cyclic groups, though again we only state it here in the torsion free case.

Proposition 10.6 ([4] Corollary 2.3) *Let H be a torsion free hyperbolic group and let us form the HNN extension $G = \langle H, t \rangle$ over the infinite cyclic subgroups $A = \langle a \rangle$ and $B = \langle b \rangle$ where $tat^{-1} = b$. Then G is word hyperbolic unless*

Either: some conjugate of B in H intersects A non trivially

Or: The intersector $I_H(a) \neq A$ and the intersector $I_H(b) \neq B$.

In both of these cases G contains a Baumslag - Solitar subgroup and so is

not word hyperbolic.

We will again need a version of Lemma 10.2 for HNN extensions (proved in the same manner using reduced forms):

Lemma 10.7 *Let $G = \langle H, t \rangle$ be the HNN extension of the torsion free hyperbolic group H over the infinite cyclic subgroups $A = \langle a \rangle$ and $B = \langle b \rangle$ where $tat^{-1} = b$. Then on taking an element $h \in H$ we have:*

- (i) *If h is not maximal in H then it is clearly not maximal in G .*
- (ii) *If h is maximal in H and is not conjugate in H into $A \cup B$ then h is still maximal in G .*

Now suppose that no conjugate of B in H intersects A non trivially. Suppose also that h is maximal in H and without loss of generality is conjugate in H into A , thus to exactly one of a and a^{-1} so say a . In particular a is maximal in H and thus G is word hyperbolic by Proposition 10.6.

- (iii) *Suppose that b is also maximal in H then a and b are maximal in G .*
- (iv) *Suppose that b is not maximal in H then clearly any element conjugate in H to $a^{\pm 1}$ is maximal in H but not in G and any element conjugate in H to $b^{\pm 1}$ is not maximal in H nor in G .*

We can now give our final result.

Theorem 10.8 *Suppose that $G(\Gamma)$ is a finite graph of groups where all edge groups are infinite cyclic and all vertex groups are torsion free word hyperbolic. Then G is word hyperbolic unless*

- (i) *there exists a complete conjugacy path in $G(\Gamma)$ or*
- (ii) *there exists a full non maximal path in $G(\Gamma)$*

in which case G contains a Baumslag - Solitar group and so is not word hyperbolic. If either of these hold then by Proposition 8.2 for (i) and Proposition 10.4 for (ii) we can assume that such a path passes through any unoriented edge at most once and so there are only finitely many of these paths to check.

Proof. This reduces to Theorem 10.5 if Γ is a tree as $G(\Gamma)$ cannot then contain a reduced closed conjugacy path. Otherwise we take a maximal tree T and we assume by induction on the number of edges in $\Gamma \setminus T$ that:

- (i) *If there exists a complete conjugacy path in $G(\Gamma)$ then G contains a Baumslag-Solitar group and so is not word hyperbolic.*
- (ii) *If there exists a full non maximal path in $G(\Gamma)$ then G contains a Baumslag - Solitar group and so is not word hyperbolic.*

- (iii) If $G(\Gamma)$ contains no full non maximal paths and no complete conjugacy paths then G is word hyperbolic.
- (iv) If G is word hyperbolic then a vertex element g_v which is maximal in its vertex group is non maximal in G if and only if there exists a semi non maximal path that ends at g_v .

Thus we assume we have the graph of groups $H(\Delta)$ satisfying these conditions and we add an edge e to Δ to obtain Γ . The base case where Γ is a tree is covered by Theorem 10.5 (or Γ is a single point whereupon everything automatically holds). In general we can assume that $H(\Delta)$ is word hyperbolic because otherwise the inductive hypothesis applied to $H(\Delta)$ tells us that H and thus G would contain a Baumslag - Solitar group. In particular there are no complete conjugacy paths or full non maximal paths in $H(\Delta)$. On adding the edge e to $H(\Delta)$ to obtain $G(\Gamma)$, where G will be the HNN extension $\langle H, t \rangle$ for $A = \langle a \rangle$ a cyclic subgroup of G_v and $B = \langle b \rangle$ of G_w , we suppose that $C = \langle c \rangle$ is the maximal cyclic subgroup of G_v containing a and $D = \langle d \rangle$ that for b in G_w .

First suppose that addition of e creates a complete conjugacy path in $G(\Gamma)$. This means that there must have been a reduced conjugacy path from a to b lying in $H(\Delta)$, or possibly that a and b lie in the same vertex group with powers conjugate in that group, so our HNN extension fails the “Either” condition for hyperbolicity in Proposition 10.6. Hence G contains a Baumslag - Solitar subgroup and so is not word hyperbolic (nor any group obtained from G by further HNN extensions). Alternatively if on addition of e we find a full non maximal path in $G(\Gamma)$, when there were none in $H(\Delta)$, then e might be labelled by arrows: if there are two arrows on e then right away G satisfies the “Or” condition in Proposition 10.6 and so contains a Baumslag - Solitar group. If e has one arrow then suppose that e joins $a = g_v \in G_v$ to $b = g_w \in G_w$ (where possibly $v = w$) and b is not maximal in G_w so our arrow on e points towards w . But now by starting at v and then following the rest of our full non maximal path, we have that the reverse of this is a semi non maximal path which must lie purely within $H(\Delta)$ and so by (iv) applied to $H(\Delta)$ we have that $a \in G_v$ is not maximal in H either and so G contains a Baumslag - Solitar group by Proposition 10.6. Similarly if e is unlabelled then we can do the same in both directions from v and from w .

For (iii), suppose that G fails to be word hyperbolic because the “Either” condition holds in Proposition 10.6. Then by Theorem 7.6 applied to $H(\Delta)$ we have that either $v = w$ and a conjugate within G_v of B meets A , in which

case the loop e itself is a complete conjugacy path, or there is a conjugacy path from $a \in G_v$ to $b \in G_w$ which can be made reduced by Proposition 7.2 and thus is made complete by adding on e .

Thus now the “Or” condition in Proposition 10.6 is the only way that G can fail to be word hyperbolic. So suppose that both a and b fail to be maximal in H . It could be that a and b are not maximal in each of their respective vertex groups, in which case e is marked with two arrows and itself immediately provides a full non maximal path in $G(\Gamma)$. Alternatively it could be that a (say) is maximal in G_v but not in H whereas b is not even maximal in G_w , whereupon we use the inductive hypothesis (iv) on $H(\Delta)$ to get us a semi non maximal path ending in $a \in G_v$, reduced without loss of generality, which results in a full non maximal path when putting e on the end. If however both a and b are maximal in their respective vertex groups then the edge e is unlabelled and we have two reduced semi non maximal paths, each in $H(\Delta)$ with one ending in $a \in G_v$ and the other in $b \in G_w$, thus by putting these together with e in the middle we again have a full non maximal path in $G(\Gamma)$.

Finally we must establish the inductive hypothesis (iv) for $G(\Gamma)$ by determining which are the “new non maximal vertex elements”. First say that both a and b are maximal in H then we have by Lemma 10.7 that there are none. But no new semi non maximal paths can be created using the edge e , which is unlabelled, because following one from its start until we reach e would produce a semi non maximal path in $H(\Delta)$ which ended in either a or b , thus by the inductive hypothesis one of these elements would be non maximal in H .

So one of a or b is non maximal in H , but both being non maximal puts us in the non word hyperbolic case, thus we now suppose without loss of generality that b is non maximal in H but a is maximal. Then again by Lemma 10.7 the new non maximal vertex elements will be those that are conjugate in H to $a^{\pm 1} \in G_v$, thus we can take our semi non maximal path in $H(\Delta)$ that ends in $b \in G_w$ and add the edge e (or just take the edge e if b is not maximal in G_w) to get one that ends in $a \in G_v$. If we further have maximal elements in H that are conjugate in H but not in G_v to a (or to a^{-1} in which case we replace every element in the path by its inverse) then we have a conjugacy path from such an element to $a \in G_v$. We can assume that no arrows appear on any edges in this path because otherwise we either lose maximality of this element in $H(\Delta)$ or we create a full non maximal path in $G(\Gamma)$, just as in the proof of Theorem 10.5. Thus by first following

the semi maximal path to b , then the reverse of e and finally the reverse of this conjugacy path to a , we obtain our semi non maximal path to any of these “new non maximal vertex elements”. Moreover this is the only way in which new semi non maximal paths can be created when moving from $H(\Delta)$ to $G(\Gamma)$ because, once we assume such a path only passes through each non oriented edge at most once, it would have to travel through e from b to a (else it would lie in $H(\Delta)$ or the element a would not be maximal in H), thus our induction is complete.

□

References

- [1] J. W. Anderson, J. Aramayona and K. J. Shackleton, *An obstruction to the strong relative hyperbolicity of a group*, J. Group Theory **10** (2007), 749–756.
- [2] G. Baumslag and R. Strebel, *Some finitely generated, infinitely related metabelian groups with trivial multiplier*, J. Algebra **40** (1976), 46–62.
- [3] M. Bestvina and M. Feighn, *A combination theorem for negatively curved groups*, J. Differential Geom. **35** (1992), 85–101.
- [4] M. Bestvina and M. Feighn, *Addendum and correction to “A combination theorem for negatively curved groups”*, J. Differential Geom. **43** (1996), 783–788.
- [5] J. O. Button and R. Kropholler, *Non hyperbolic free-by-cyclic and one-relator groups*, <http://arxiv.org/1503.01989>
- [6] C. Champetier and V. Guirardel, *Limit groups as limits of free groups*, Israel J. Math. **146** (2005), 1–75.
- [7] T. Hsu and D. T. Wise, *Cubulating graphs of free groups with cyclic edge groups*, Amer. J. Math. **132** (2010), 1153–1188.
- [8] A. Karrass and D. Solitar, *The free product of two groups with a malnormal amalgamated subgroup*, Canad. J. Math. **23** (1971), 933–959.

- [9] P. H. Kropholler, *A note on centrality in 3-manifold groups*, Math. Proc. Cambridge Philos. Soc. **107** (1990), 261–266.
- [10] P. H. Kropholler, *Baumslag-Solitar groups and some other groups of cohomological dimension two*, Comment. Math. Helvetici **65** (1990), 547–558.
- [11] G. Levitt, *Quotients and subgroups of Baumslag-Solitar groups*, J. Group Theory **18** (2015), 1–43.
- [12] K. I. Lossov, *SQ-universality of free products with amalgamated finite subgroups*, Sibirsk. Mat. Zh. **27** (1986), 128–139, 225.
- [13] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, Dover Publications, Inc., Mineola, New York 2004.
- [14] A. Minasyan and D. Osin, *Acylindrical hyperbolicity of groups acting on trees*, Math. Ann. **362** (2015), 1055–1105.
- [15] D. L. Moldavanskii, *On the intersection of subgroups of finite index in the Baumslag-Solitar groups*, Math. Notes **87** (2010), 88–95.
- [16] D. Osin, *Relatively hyperbolic groups: Intrinsic geometry, algebraic properties, and algorithmic problems*. *Memoirs AMS* **179** (2006), no. 843.
- [17] D. T. Wise, *Subgroup separability of graphs of free groups with cyclic edge groups*, Quart. J. Math. **51** (2000), 107–129.
- [18] D. T. Wise, *From Riches to Raags: 3-Manifolds, Right-Angled Artin Groups, and Cubical Geometry*, CBMS Regional Conference Series in Mathematics No. 117, American Mathematical Society, Providence, RI, 2012.

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